Chapter 12

The second derivative and applications

Objectives

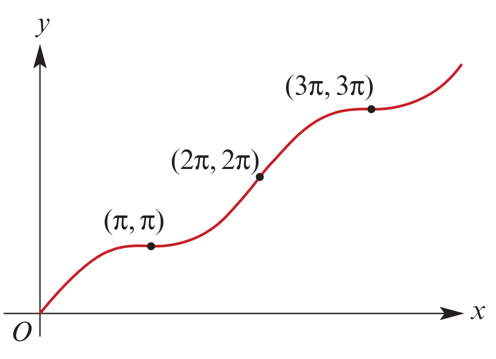
- To use the notation for the **second derivative** of a function.
- To recognise **acceleration** as the second derivative of position with respect to time.
- To use the second derivative in graph sketching:
  - investigate **concavity** and **points of inflection**
  - apply the **second derivative test** to determine the nature of stationary points.
- To solve **optimisation problems**.

In this chapter, we introduce the **second derivative** of a function. This is simply the derivative of the derivative. We will see that the second derivative can provide extra information about the shape of the graph of a function.

For example, part of the graph of $y = x + \sin x$ is shown for $x \geq 0$. You can clearly see how the gradient is changing.

From the point $(0, 0)$ to the point (π, π), the gradient is decreasing. At (π, π), this changes and the gradient starts increasing. At $(2\pi, 2\pi)$, the gradient starts decreasing again, and so on.

These points are called **points of inflection**. They can be identified through the second derivative.



This chapter covers Unit 4 Topic 1: Further differentiation and applications 3.

12A The second derivative and acceleration

For the function f with rule $f(x)$, the derivative is denoted by f' and has rule $f'(x)$. This notation is extended to taking the derivative of the derivative: the new function is denoted by f'' and has rule $f''(x)$. This new function is known as the **second derivative**.

For example, consider the function g with rule $g(x) = 2x^3 - 4x^2$.

- The derivative has rule $g'(x) = 6x^2 - 8x$.
- The second derivative has rule $g''(x) = 12x - 8$.

Note: The second derivative might not exist at a point even if the first derivative does.
For example, let $f(x) = x^{\frac{4}{3}}$. Then $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$ and $f''(x) = \frac{4}{9}x^{-\frac{2}{3}}$.
We see that $f'(0) = 0$, but the second derivative $f''(x)$ is not defined at $x = 0$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Example 1

Find the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$ **b** $y = e^x \sin x$

Solution

a
$$f(x) = 6x^4 - 4x^3 + 4x$$
$$f'(x) = 24x^3 - 12x^2 + 4$$
$$f''(x) = 72x^2 - 24x$$

b
$$y = e^x \sin x$$
$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \quad \text{(by the product rule)}$$
$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$
$$= 2e^x \cos x$$

Example 2

If $f(x) = e^{2x}$, find $f''(0)$.

Solution

$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x}$$
$$f''(x) = 4e^{2x}$$

Therefore $f''(0) = 4e^0 = 4$.

Example 3

Consider $f(x) = x^3 - 2x^2 + 4x - 6$.

a Find $f''(x)$.

b Solve the equation $f''(x) = 0$ for x.

Solution

a
$$f(x) = x^3 - 2x^2 + 4x - 6$$
$$f'(x) = 3x^2 - 4x + 4$$
$$f''(x) = 6x - 4$$

b
$$f''(x) = 0$$
$$6x - 4 = 0$$
$$\therefore \quad x = \frac{2}{3}$$

Example 4

Consider $y = x^2e^x$.

a Find $\dfrac{d^2y}{dx^2}$.

b Solve the equation $\dfrac{d^2y}{dx^2} = 0$ for x.

Solution

a
$$y = x^2e^x$$
$$\frac{dy}{dx} = 2xe^x + x^2e^x$$
$$\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2e^x$$
$$= 2e^x + 4xe^x + x^2e^x$$

b $\dfrac{d^2y}{dx^2} = 0$ implies
$$2e^x + 4xe^x + x^2e^x = 0$$
$$e^x(2 + 4x + x^2) = 0$$
$$x^2 + 4x + 2 = 0$$

Therefore $x = -2 + \sqrt{2}$ or $x = -2 - \sqrt{2}$.

Motion in a straight line

We have studied motion in a straight line in Sections 8J, 9G and 10G.

Recall that an object's position, x m, at time t seconds is specified with respect to a reference point O on the line. Velocity, v m/s, and acceleration, a m/s^2, are given by:

$$\text{velocity} \quad v = \frac{dx}{dt} \qquad \text{acceleration} \quad a = \frac{dv}{dt}$$

We can now recognise that acceleration is the second derivative of position with respect to time:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

That is, the object's acceleration is the rate of change of the rate of change of its position.

Example 5

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by $x = 5 + \sin(2\pi t)$ for $0 \le t \le 2$. Find:

a at what times and in what positions the particle will have zero velocity

b its acceleration at those instants.

Solution

a Velocity $v = \dfrac{dx}{dt} = 2\pi\cos(2\pi t)$

Solve the equation $v = 0$ for $0 \le t \le 2$:

$$\begin{aligned} 2\pi\cos(2\pi t) &= 0 \\ \cos(2\pi t) &= 0 \\ 2\pi t &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \\ \therefore \quad t &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4} \text{ or } \frac{7}{4} \end{aligned}$$

The times and positions at which the velocity is zero:

- $t = \frac{1}{4}$, $x = 5 + \sin\left(\frac{\pi}{2}\right) = 6$ m
- $t = \frac{3}{4}$, $x = 5 + \sin\left(\frac{3\pi}{2}\right) = 4$ m
- $t = \frac{5}{4}$, $x = 5 + \sin\left(\frac{5\pi}{2}\right) = 6$ m
- $t = \frac{7}{4}$, $x = 5 + \sin\left(\frac{7\pi}{2}\right) = 4$ m

b Acceleration $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = -4\pi^2\sin(2\pi t)$

The acceleration when the velocity is zero:

- $t = \frac{1}{4}$, $a = -4\pi^2\sin\left(\frac{\pi}{2}\right) = -4\pi^2$ m/s^2
- $t = \frac{3}{4}$, $a = -4\pi^2\sin\left(\frac{3\pi}{2}\right) = 4\pi^2$ m/s^2
- $t = \frac{5}{4}$, $a = -4\pi^2\sin\left(\frac{5\pi}{2}\right) = -4\pi^2$ m/s^2
- $t = \frac{7}{4}$, $a = -4\pi^2\sin\left(\frac{7\pi}{2}\right) = 4\pi^2$ m/s^2

Section summary

- The **second derivative** of a function f is the derivative of the derivative of f.
- For a function f with rule $f(x)$, the second derivative of f is denoted by f'' and has rule $f''(x)$.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.
- For an object moving in a straight line with position x at time t:
  - velocity $v = \dfrac{dx}{dt}$
  - acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

Exercise 12A

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Example 1

1 Find the second derivative of each of the following:

a $2x + 5$ **b** x^8 **c** $\sqrt{x}$ **d** $(2x+1)^4$ **e** $\sin x$

f $\cos x$ **g** e^x **h** $\ln x$ **i** $\dfrac{1}{x+1}$ **j** $\sin\left(2x + \dfrac{\pi}{4}\right)$

2 Find the second derivative of each of the following:

a $\sqrt{x^5}$ **b** $(x^2+3)^4$ **c** $\sin\left(\dfrac{x}{2}\right)$

d $3\cos(4x+1)$ **e** $\dfrac{1}{2}e^{2x+1}$ **f** $\ln(2x+1)$

g $x^4 + 3x^2 - 7x + 2$ **h** x^3e^x **i** $x\ln x$

3 For each of the following, find $f''(x)$:

a $f(x) = 6e^{3-2x}$ **b** $f(x) = -8e^{-0.5x^2}$ **c** $f(x) = e^{\ln x}$

d $f(x) = \ln(x^2 + 2x)$ **e** $f(x) = 2(1-3x)^5$ **f** $f(x) = e^{x^2}$

g $f(x) = \dfrac{x-1}{x+1}$ **h** $f(x) = \dfrac{1}{\sqrt{1-x}}$ **i** $f(x) = 5\sin(3-x)$

j $f(x) = \cos(1-3x)$ **k** $f(x) = \sin\left(\dfrac{x}{3}\right)$ **l** $f(x) = \cos\left(\dfrac{x}{4}\right)$

Example 2

4 For each of the following, find $f''(0)$:

a $f(x) = e^{\sin x}$ **b** $f(x) = e^{-\frac{1}{2}x^2}$ **c** $f(x) = \sqrt{1-x^2}$ **d** $f(x) = \cos(x^2)$

Example 3

5 For each of the following, solve the equation $f''(x) = 0$ for x:

a $f(x) = 2x^3 + 4x^2$ **b** $f(x) = 5 - x - x^2 + 5x^3$ **c** $f(x) = x^4 - 3x^2 - 4x$

Example 4

6 For each of the following, solve the equation $\dfrac{d^2y}{dx^2} = 0$ for x:

a $y = 2xe^x$ **b** $y = x^2e^x - xe^x$ **c** $y = x^3e^x$

d $y = \dfrac{x}{\ln x}$ **e** $y = \dfrac{\ln x}{x}$

CF

Example 5

7 A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by $x = 7 + 2\cos\left(\dfrac{\pi t}{4}\right)$ for $0 \le t \le 8$. Find:

a at what times and in what positions the particle has zero velocity

b its acceleration at those instants.

8 A particle moves along the x-axis so that at time t seconds its distance, x m, from the origin O is given by $x = e^{-t} - e^{-2t}$ for $t \ge 0$.

a Find the particle's velocity and acceleration at time t seconds.

b Find its maximum distance from the origin and the time at which this occurs.

c Find the acceleration of the particle when it reaches this maximum distance.

d Find the maximum speed of the particle as it returns towards O.

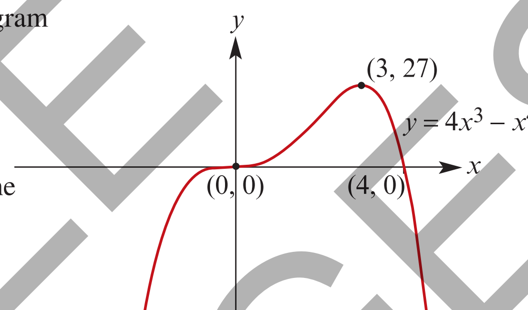
12B Using the second derivative in graph sketching

In Section 8L, you have used the first derivative when sketching the graphs of polynomial and other functions. The second derivative enables us to find out more information about these graphs. We start this section by considering the graph of $y = 4x^3 - x^4$.

▶ The graph of $y = 4x^3 - x^4$

The graph of this function is shown in the diagram on the right.

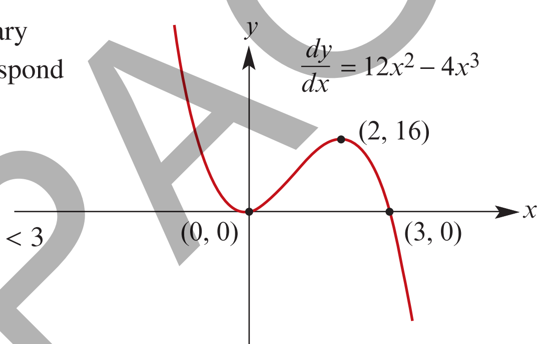
There is a local maximum at $(3, 27)$ and a stationary point of inflection at $(0, 0)$. These points have been determined by considering the derivative function $\frac{dy}{dx} = 12x^2 - 4x^3$.



The graph of the derivative function

Note that the local maximum and the stationary point of inflection of the original graph correspond to the x-axis intercepts of the graph of the derivative.

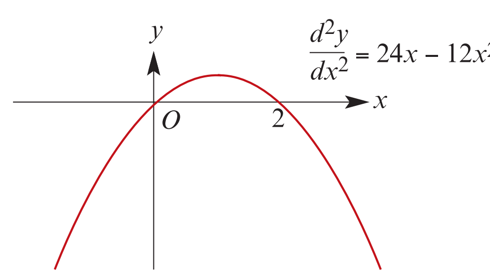
Also, it can be seen that the gradient of the original graph is positive for $x < 0$ and $0 < x < 3$ and negative for $x > 3$.



The graph of the second derivative function

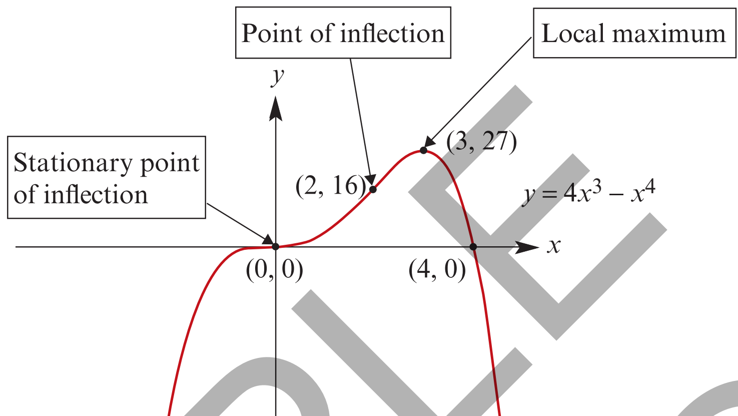
Further information can be obtained by considering the graph of the second derivative.

The graph of the second derivative reveals that, at the points on the original graph where $x = 0$ and $x = 2$, there are important changes in the gradient.



- At the point where $x = 0$, the gradient of $y = 4x^3 - x^4$ changes from decreasing (positive) to increasing (positive). This point is also a stationary point, but it is neither a local maximum nor a local minimum. It is known as a **stationary point of inflection**.
- At the point where $x = 2$, the gradient of $y = 4x^3 - x^4$ changes from increasing (positive) to decreasing (positive). This point is called a **point of inflection**. In this case, the point corresponds to a local maximum of the derivative graph.

The gradient of $y = 4x^3 - x^4$ increases on the interval $(0, 2)$ and then decreases on the interval $(2, 3)$. The point $(2, 16)$ is the point of maximum gradient of $y = 4x^3 - x^4$ for the interval $(0, 3)$.



Behaviour of tangents

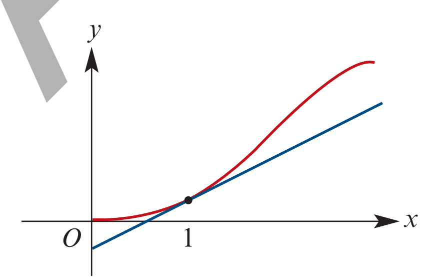
A closer look at the graph of $y = 4x^3 - x^4$ for the interval $(0, 3)$ and, in particular, the behaviour of the tangents to the graph in this interval will reveal more.

The tangents at $x = 1$, 2 and 2.5 have equations $y = 8x - 5$, $y = 16x - 16$ and $y = \frac{25}{2}x - \frac{125}{16}$ respectively. The following graphs illustrate the behaviour.

- The first diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 1$.

  The tangent lies *below* the graph in the immediate neighbourhood of where $x = 1$.

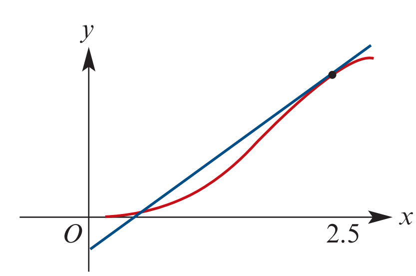
  For the interval $(0, 2)$, the gradient of the graph is increasing; the graph is said to be *concave up*.



- The second diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2.5$.

  The tangent lies *above* the graph in the immediate neighbourhood of where $x = 2.5$.

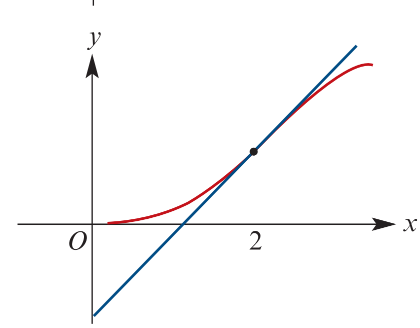
  For the interval $(2, 3)$, the gradient of the graph is decreasing; the graph is said to be *concave down*.



- The third diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2$.

  The tangent *crosses* the graph at the point $(2, 16)$.

  At $x = 2$, the gradient of the graph changes from increasing to decreasing; the point $(2, 16)$ is said to be a *point of inflection*.



Concavity and points of inflection

For a curve $y = f(x)$:

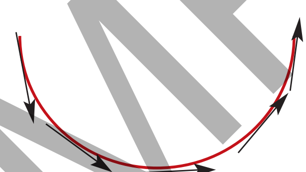
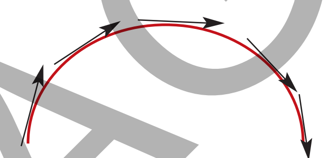
- the derivative $f'(a)$ gives the gradient of the curve at $x = a$
- the second derivative $f''(a)$ gives the rate of change of the gradient of the curve at $x = a$.

We have met the ideas of concave up and concave down in the example at the beginning of this section. We now give the definitions of these ideas.

Concave up and concave down

For a curve $y = f(x)$:

- If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
- If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.

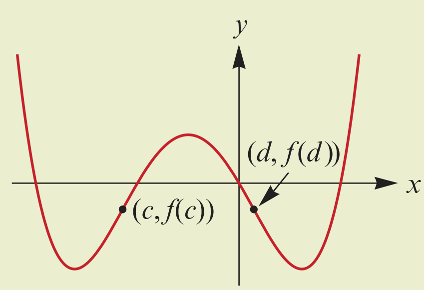
Concave up for an interval	Concave down for an interval
 The tangent is below the curve at each point and the gradient is increasing i.e. $f''(x) > 0$	 The tangent is above the curve at each point and the gradient is decreasing i.e. $f''(x) < 0$

Point of inflection

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**.

That is, a point of inflection occurs where the sign of the second derivative changes.

In the graph on the right, there are points of inflection at $x = c$ and $x = d$.



Note: At a point of inflection, the tangent will pass through the curve.

At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.

For example, consider $f(x) = x^4$. Then $f''(x) = 12x^2$ and so $f''(0) = 0$. But the graph of $y = x^4$ has a local minimum at $x = 0$.

From now on, we can use these new ideas in our graphing.

Example 6

For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

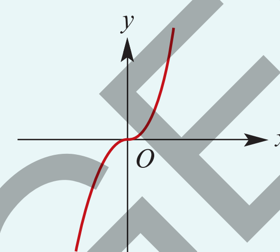
a $f(x) = x^3$

b $f(x) = -x^3$

c $f(x) = x^3 - 3x^2 + 1$

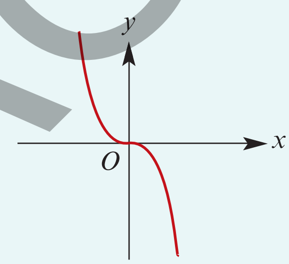
Solution

a
- There is a stationary point of inflection at $(0, 0)$. At $x = 0$, the gradient is zero and the curve changes from concave down to concave up.
- The curve is concave up on the interval $(0, \infty)$. The second derivative is positive on this interval.



Note: The tangent at $x = 0$ is the line $y = 0$.

b
- There is a stationary point of inflection at $(0, 0)$. At $x = 0$, the gradient is zero and the curve changes from concave up to concave down.
- The curve is concave up on the interval $(-\infty, 0)$. The second derivative is positive on this interval.



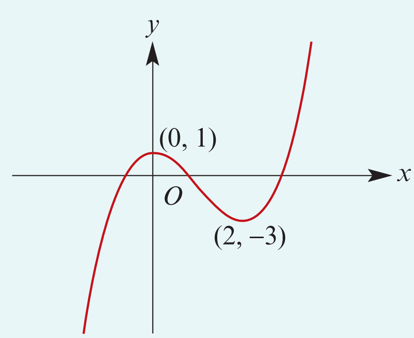
Note: The tangent at $x = 0$ is the line $y = 0$.

c
$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x$$
$$f''(x) = 6x - 6$$

There is a local maximum at $(0, 1)$ and a local minimum at $(2, -3)$.

The second derivative is zero at $x = 1$, it is positive for $x > 1$, and it is negative for $x < 1$.

- There is a point of inflection at $(1, -1)$.
- The curve is concave up on the interval $(1, \infty)$.



Test for local maximum or minimum

The following test provides a useful method for identifying local maximums and minimums.

Second derivative test

For the graph of $y = f(x)$:

- If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum, as the curve is concave up.
- If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum, as the curve is concave down.
- If $f''(a) = 0$, then further investigation is necessary.

Example 7

Consider the graph of $y = f(x)$, where $f(x) = x^2(10 - x)$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$.

Solution

We have $f(x) = x^2(10 - x) = 10x^2 - x^3$

$f'(x) = 20x - 3x^2$

$f''(x) = 20 - 6x$

a $f'(x) = 0$ implies $x(20 - 3x) = 0$, and therefore $x = 0$ or $x = \frac{20}{3}$.

Since $f''(0) = 20 > 0$, there is a local minimum at $(0, 0)$.

Since $f''\left(\frac{20}{3}\right) = -20 < 0$, there is a local maximum at $\left(\frac{20}{3}, \frac{4000}{27}\right)$.

b $f''(x) = 0$ implies $x = \frac{10}{3}$.

We have $f''(x) > 0$ for $x < \frac{10}{3}$

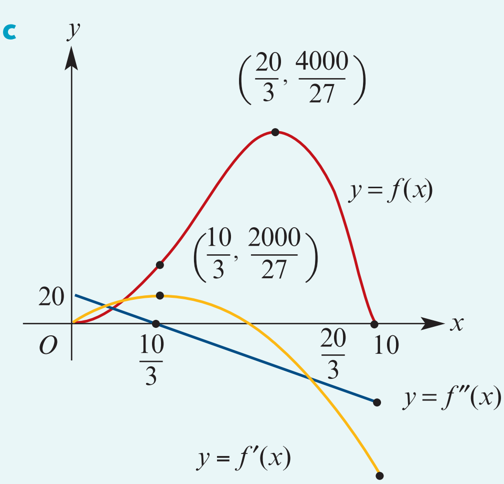
and $f''(x) < 0$ for $x > \frac{10}{3}$.

Hence there is a point of inflection at $\left(\frac{10}{3}, \frac{2000}{27}\right)$.

The gradient at this point is $\frac{100}{3}$.

Note: The maximum gradient of $y = f(x)$ is at the point of inflection.

c



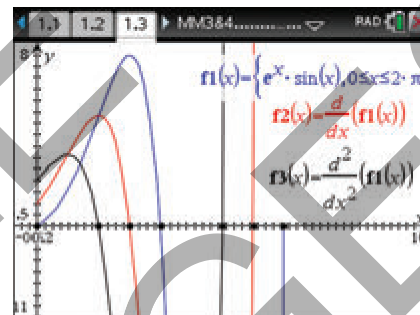
Example 8

Using a graphics calculator, find approximate coordinates for the stationary points and the points of inflection on the graph of the function

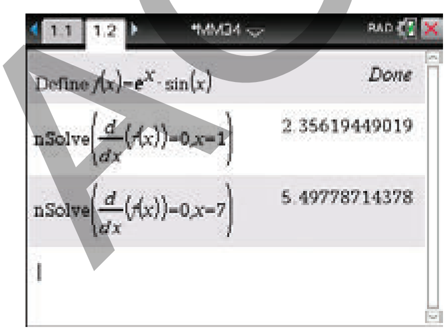
$$f(x) = e^x \sin x, \quad x \in [0, 2\pi]$$

Using the TI-Nspire CX non-CAS

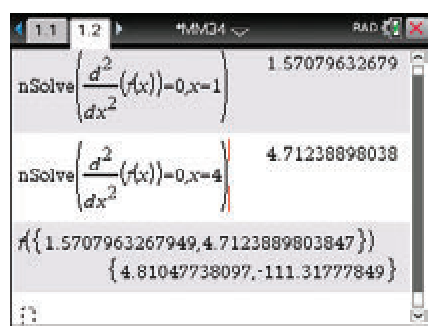
- Plot the graphs of the original function ($f1$), its derivative ($f2$) and its second derivative ($f3$) as shown. Note that:
  - The graph of the derivative ($f2$) has two x-axis intercepts, so there are two stationary points.
  - The graph of the second derivative ($f3$) crosses the x-axis twice, so there are two points of inflection.



- In a **Calculator** application, define the function $f(x) = e^x \sin(x)$.
- Use **nSolve** to find the x-values such that the derivative equals zero, for $x \in [0, 2\pi]$.



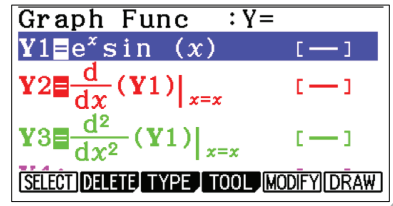
- Use **nSolve** to find the x-values such that the second derivative equals zero, for $x \in [0, 2\pi]$.
- Substitute the x-values in $f(x)$ to find the corresponding y-values:
  - The stationary points are at $(2.36, 7.46)$ and $(5.50, -172.64)$.
  - The points of inflection are at $(1.57, 4.81)$ and $(4.71, -111.32)$.



Notes:

- The derivative templates can be accessed from the 2D-template palette.
- In the **Graphs** application, you can use menu > **Trace** > **Graph Trace** to find the x-axis intercepts of the derivative and the second derivative.
- When using **nSolve**, you will need to try several guess values in order to find all the solutions.

Using the Casio



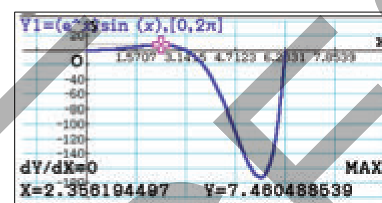
- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = e^x \sin(x)$ in $Y1$.
- Enter the derivative of $Y1$ in $Y2$:

  (OPTN) (F2) (F1) (F1) (1) ► (X,θ,T) (EXE)

- Enter the second derivative of $Y1$ in $Y3$:

  (OPTN) (F2) (F2) (F1) (1) ► (X,θ,T) (EXE)

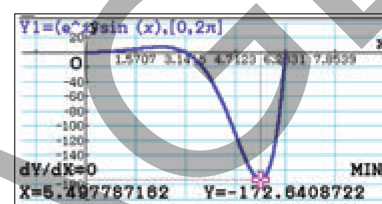
Turning points



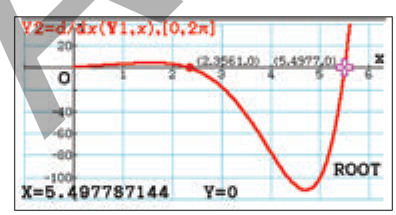
- Plot the graph of the original function $Y1$:

  ▲ (F1) ▲ (F1) (F6)

- Go to **G-Solve** (SHIFT) (F5); select **Maximum** (F2). The local maximum is at (2.36, 7.46).

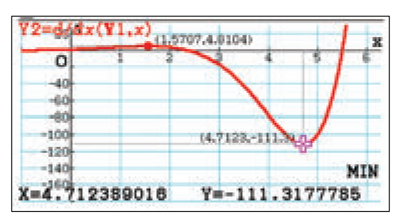


- Go to **G-Solve** (SHIFT) (F5); select **Minimum** (F3). The local minimum is at (5.50, −172.64).



Note: The x-coordinates of the stationary points are the x-axis intercepts of the graph of the derivative function $Y2$.

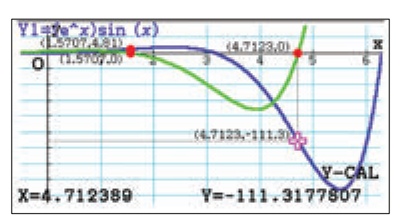
Points of inflection



- Press (EXIT) to return to the function list.
- Plot the graph of the derivative function $Y2$:

  ▲ (F1) ▲ (F1) (F6)

- Find the turning points using **Maximum** and **Minimum** from the **G-Solve** menu.
- The points of inflection occur when $x = 1.571$ and $x = 4.712$.



- On the graph of the original function $Y1$, find the y-coordinate of each inflection point using **y-Cal** from the **G-Solve** menu.
- The points of inflection are at (1.57, 4.81) and (4.71, −111.32).

Note: The x-coordinates of the points of inflection can also be found by determining where the graph of the second derivative function $Y3$ crosses the x-axis.

Example 9

Sketch the graph of $f(x) = x^4 - 8x^3 + 18x^2 + 4$, locating the stationary points and the points of inflection.

Solution

$f(x) = x^4 - 8x^3 + 18x^2 + 4$

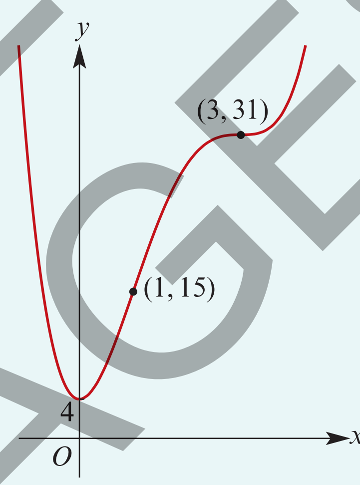
$f'(x) = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$

$f''(x) = 12x^2 - 48x + 36 = 12(x-1)(x-3)$

Stationary points

$f'(x) = 0$ implies $x = 0$ or $x = 3$

- Since $f''(0) = 36 > 0$, there is a local minimum at $(0, 4)$.
- Since $f''(3) = 0$, the test is inconclusive; further investigation is required.



Points of inflection

$f''(x) = 0$ implies $x = 1$ or $x = 3$

When $x < 1$, $f''(x) > 0$.
When $1 < x < 3$, $f''(x) < 0$.
When $x > 3$, $f''(x) > 0$.

- There is a point of inflection at $(1, 15)$ and a stationary point of inflection at $(3, 31)$.

Section summary

- For a curve $y = f(x)$:
  - If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve is increasing over the interval (a, b). The curve is said to be **concave up**.
  - If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve is decreasing over the interval (a, b). The curve is said to be **concave down**.
- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test

  For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

Exercise 12B

Skillsheet

1 Sketch a small portion of a continuous curve around a point $x = a$ having the property:

a $\frac{dy}{dx} > 0$ when $x = a$ and $\frac{d^2y}{dx^2} > 0$ when $x = a$

b $\frac{dy}{dx} < 0$ when $x = a$ and $\frac{d^2y}{dx^2} < 0$ when $x = a$

c $\frac{dy}{dx} > 0$ when $x = a$ and $\frac{d^2y}{dx^2} < 0$ when $x = a$

d $\frac{dy}{dx} < 0$ when $x = a$ and $\frac{d^2y}{dx^2} > 0$ when $x = a$

Example 6

2 For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

a $f(x) = x^3 - x$ **b** $f(x) = x^3 - x^2$ **c** $f(x) = x^2 - x^3$ **d** $f(x) = x^4 - x^3$

Example 7

3 Let $f(x) = \frac{x^2}{10}(20 - x)$ for $x \in [0, 20]$.

a Find the coordinates of the stationary points and determine their nature using the second derivative test.

b Find the coordinates of the point of inflection and find the gradient at this point.

c On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 20]$.

4 Let $f(x) = 2x^3 + 6x^2 - 12$ for $x \in \mathbb{R}$.

a **i** Find $f'(x)$. **ii** Find $f''(x)$.

b Find the coordinates of the stationary points and use the second derivative test to establish their nature.

c Use $f''(x)$ to find the point of inflection.

5 Repeat Question 4 for each of the following functions:

a $f(x) = \sin x,\ x \in [0, 2\pi]$ **b** $f(x) = xe^x,\ x \in \mathbb{R}$

Example 9

6 Sketch the graph of $y = x^3(4 - x)$, locating the stationary points and the points of inflection.

7 Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$, locating the stationary points and the points of inflection.

8 Let $f(x) = x(10 - x)e^x$ for $x \in [0, 10]$.

a Find $f'(x)$ and $f''(x)$.

b Sketch the graphs of $y = f(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 10]$.

c Find the value of x for which the gradient of the graph of $y = f(x)$ is a maximum and indicate this point on the graph of $y = f(x)$.

SF

CF

SF

9 Find the coordinates of the points of inflection of $y = x - \sin x$ for $x \in [0, 4\pi]$.

10 For each of the following functions, find the values of x in the interval $[0, 2\pi]$ for which the graph of the function has a point of inflection:

a $y = \sin x$ **b** $y = \tan x$ **c** $y = \sin(2x)$

CF

11 Show that the parabola with equation $y = ax^2 + bx + c$ has no points of inflection.

12 For the curve with equation $y = 2x^3 - 9x^2 + 12x + 8$, find the values of x for which:

a $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **b** $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

SF

13 For each of the following functions, determine the coordinates of any points of inflection and the gradient of the graph at these points:

a $y = x^3 - 6x$ **b** $y = x^4 - 6x^2 + 4$ **c** $y = 3 - 10x^3 + 10x^4 - 3x^5$

d $y = (x^2 - 1)(x^2 + 1)$ **e** $y = x\sqrt{x + 1}$ **f** $y = \frac{2x}{x^2 + 1}$

14 Determine the values of x in $[-2\pi, 2\pi]$ for which the graph of $y = e^{-x} \sin x$ has:

a stationary points **b** points of inflection.

CF

15 Given that $f(x) = x^3 + bx^2 + cx$ and $b^2 > 3c$, prove that:

a the graph of f has two stationary points

b the graph of f has one point of inflection

c the point of inflection is the midpoint of the interval joining the stationary points.

16 Consider the function with rule $f(x) = 2x^2 \ln(x)$.

a Find $f'(x)$.

b Find $f''(x)$.

c Find the stationary points and the points of inflection of the graph of $y = f(x)$.

17 The graph of $y = x^3 - ax^2 + bx + c$ passes through the point $(2, 7)$, has a local maximum when $x = 1$ and a point of inflection when $x = 3$.

a Find the values of a, b and c.

b Sketch the graph.

18 The graph of $y = ax^3 + bx + c$ has a point of inflection where $y = 3$, a local maximum where $x = 1$ and passes through the point $(2, 1)$.

a Find the values of a, b and c.

b Give the value of x for which there is a local minimum.

c Sketch the graph.

12C Absolute maximum and minimum values

Local maximum and minimum values were considered in the previous section. These are often not the actual maximum and minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

More precisely, for a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the absolute maximum value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the absolute minimum value of the function.

Example 10

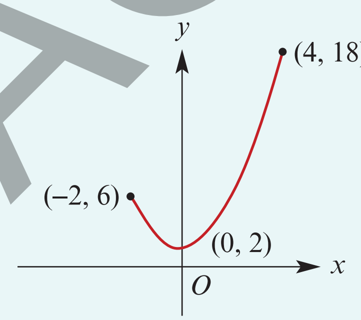
Let $f(x) = x^2 + 2$ for $x \in [-2, 4]$. Find the absolute maximum value and the absolute minimum value of the function.

Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

(Note that the absolute minimum occurs at a stationary point of the graph. The absolute maximum occurs at an endpoint, not at a stationary point.)



Example 11

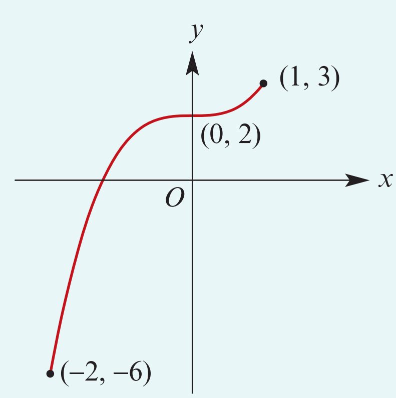
Let $f(x) = x^3 + 2$ for $x \in [-2, 1]$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

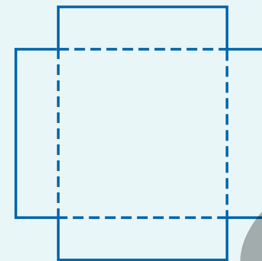
The minimum value is −6 and occurs when $x = -2$.

(Note that the absolute maximum and minimum values do not occur at stationary points.)



Example 12

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d If the height of the box must be less than 0.3 m, i.e. $x \leq 0.3$, what will be the maximum volume of the box?

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2-2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

b Let $V(x) = 4x^3 - 8x^2 + 4x$. A local maximum will occur when $V'(x) = 0$.
We have $V'(x) = 12x^2 - 16x + 4$, and so $V'(x) = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x-1)(x-1) &= 0 \\ \therefore \quad x = \frac{1}{3} \text{ or } x &= 1 \end{aligned}$$

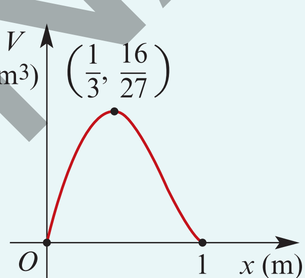
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

x		$\frac{1}{3}$		1
$V'(x)$	+	0	−	0
shape of V	/	—	\	—

The maximum occurs when $x = \frac{1}{3}$.

$$\therefore \text{ Maximum volume} = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} = \frac{16}{27} \text{ m}^3$$

c



d The local maximum of $V(x)$ defined on $[0, 1]$ is at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But $\frac{1}{3}$ is not in the interval $[0, 0.3]$.

Since $V'(x) > 0$ for all $x \in [0, 0.3]$, the maximum volume for this situation occurs when $x = 0.3$ and is 0.588 m^3.

Section summary

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \le M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \ge N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Exercise 12C

Skillsheet
Example 10

1 Let $f(x) = 2 - 8x^2$ for $x \in [-3, 3]$. Find the absolute maximum value and the absolute minimum value of the function.

SF

Example 11

2 Let $f(x) = x^3 + 2x + 3$ for $x \in [-3, 2]$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

3 Let $f(x) = 2x^3 - 6x^2$ for $x \in [-1.5, 2.5]$. Find the absolute maximum and absolute minimum values of the function.

4 Let $f(x) = 2x^4 - 8x^2$ for $x \in [-2, 6]$. Find the absolute maximum and absolute minimum values of the function.

Example 12

CF

5 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, of the block is given by $V = 15x^2 - 12x^3$.

b Find $\dfrac{dV}{dx}$.

c Find the coordinates of the local maximum of the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

6 Variables x, y and z are such that $x + y = 30$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the absolute maximum and absolute minimum values of z.

7 Consider the function $f(x) = \dfrac{1}{x-1} + \dfrac{1}{4-x}$, $x \in [2, 3]$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

8 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Find $\frac{dA}{dx}$.

c Find the value of x that makes A a minimum.

d If two squares are formed but $x \in [0, 1]$, find the maximum possible combined area of the two squares.

9 Find the maximum and minimum values of the function $g(x) = x + \frac{1}{x-2}$, $x \in [2.1, 8]$.

10 Consider the function $f(x) = \frac{1}{x+1} + \frac{1}{4-x}$, $x \in [0, 3]$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

11 Let $f(x) = \sin(2x)$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{8}\right]$. State the absolute maximum and minimum values of the function.

12 Let $f(x) = \cos(2x)$ for $x \in \left[0, \frac{\pi}{8}\right]$. State the absolute maximum and minimum values of the function.

13 Let $f(x) = 2 - x^{\frac{2}{3}}$ for $x \in [-1, 8]$. Sketch the graph and state the absolute maximum and minimum values of the function.

14 Let $f(x) = 2e^x + e^{-x}$ for $x \in [-1, 2]$. Sketch the graph and state the absolute maximum and minimum values of the function.

15 Let $f(x) = (x - 5)\ln\left(\frac{x-5}{10}\right)$ for $x \in [6, 10]$. Sketch the graph and state the absolute maximum and minimum values of the function.

12D Optimisation problems

Many practical problems require that some quantity (for example, cost of manufacture or fuel consumption) be **minimised**, that is, made as small as possible. Other problems require that some quantity (for example, profit on sales or attendance at a concert) be **maximised**, that is, made as large as possible. In both cases, we say that the quantity is to be **optimised**. We can use differential calculus to solve many of these types of problems.

We now have two methods for identifying whether a stationary point corresponds to a local maximum or local minimum value:

Method 1 create a gradient chart

Method 2 use the second derivative test.

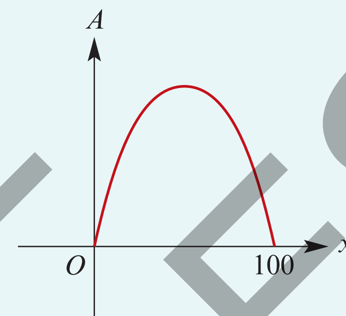
Example 13

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution

Let the length of the rectangle be x metres. Then the width is $100 - x$ metres and the area is A m^2, where

$$A = x(100 - x)$$
$$= 100x - x^2$$



The maximum value of A occurs when $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 100 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x = 50$$

From the gradient chart, the maximum area occurs when $x = 50$.

x		50	
$\frac{dA}{dx}$	+	0	−
shape of A	/	—	\

The pen with maximum area has dimensions 50 m by 50 m, and so has area 2500 m^2.

Example 14

Two variables x and y are such that $x^4 y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value. Use the second derivative test to show that this value of z is a minimum.

Solution

Obtain y in terms of x from the equation $x^4 y = 8$:

$$y = 8x^{-4}$$

Substitute in the equation $z = x + y$:

$$z = x + 8x^{-4}$$

Now z is expressed in terms of one variable, x. Differentiate with respect to x:

$$\frac{dz}{dx} = 1 - 32x^{-5}$$

A stationary point occurs where $\frac{dz}{dx} = 0$:

$$1 - 32x^{-5} = 0$$
$$32x^{-5} = 1$$
$$x^5 = 32$$
$$\therefore \quad x = 2$$

There is a stationary point at $x = 2$. The corresponding value of y is $8 \times 2^{-4} = \frac{1}{2}$.

So the corresponding value of z is

$$z = x + y = 2\tfrac{1}{2}$$

Second derivative test:

When $x = 2$, we have

$$\frac{d^2z}{dx^2} = 160x^{-6} = \frac{160}{2^6} > 0$$

which corresponds to a local minimum.

The minimum value of z is $2\frac{1}{2}$ and occurs when $x = 2$ and $y = \frac{1}{2}$.

Example 15

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution

Let the radius of the circular end of the tin be r cm, let the height of the tin be h cm and let the volume of the tin be V cm^3.

Obtain equations for the surface area and the volume.

Surface area: $100 = 2\pi r^2 + 2\pi rh$ (1)

Volume: $V = \pi r^2 h$ (2)

The process we follow now is very similar to Example 14. Obtain h in terms of r from equation (1):

$$h = \frac{1}{2\pi r}(100 - 2\pi r^2)$$

Substitute in equation (2):

$$V = \pi r^2 \times \frac{1}{2\pi r}(100 - 2\pi r^2)$$

$$\therefore \quad V = 50r - \pi r^3 \qquad (3)$$

A stationary point of the graph of $V = 50r - \pi r^3$ occurs when $\frac{dV}{dr} = 0$.

$$\frac{dV}{dr} = 0 \text{ implies } 50 - 3\pi r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{50}{3\pi}} \approx \pm 2.3$$

But $r = -2.3$ does not fit the practical situation.

Substitute $r = 2.3$ in equation (3) to find $V \approx 76.78$.

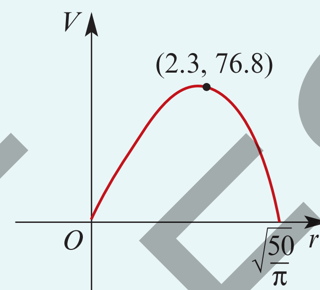
So there is a stationary point at $(2.3, 76.8)$.

Use a gradient chart to determine the nature of this stationary point.

The maximum volume is 76.78 cm^3 correct to two decimal places.

r		2.3	
$\dfrac{dV}{dr}$	+	0	−
shape of V	╱	—	╲

It can be observed that the volume is given by a function f with rule $f(r) = 50r - \pi r^3$ and domain $\left[0, \sqrt{\frac{50}{\pi}}\right]$, giving the graph on the right.



If the greatest radius the canister can have is 2 cm, then the function f has domain $[0, 2]$. It has been seen that $f'(r) > 0$ for all $r \in [0, 2]$. The maximum value occurs when $r = 2$. The maximum volume in this case is $f(2) = 100 - 8\pi \approx 74.87$ cm^3.

In some situations the variables may not be continuous. For instance, one of them may only take integer values. In such cases, we may be able to model the non-continuous case with a continuous function so that the techniques of differential calculus can be used.

Example 16

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution

Let x denote the monthly fee. Then the number of subscribers is $1000 + 100\left(\dfrac{5-x}{0.1}\right)$.

(Note that we are treating a discrete situation with a continuous function.)

Let R denote the revenue. Then

$$\begin{aligned} R &= x(1000 + 1000(5 - x)) \\ &= 1000(6x - x^2) \\ \therefore \quad \frac{dR}{dx} &= 1000(6 - 2x) \end{aligned}$$

Thus $\dfrac{dR}{dx} = 0$ implies $6 - 2x = 0$ and hence $x = 3$.

Second derivative test:

When $x = 3$, we have

$$\frac{d^2R}{dx^2} = -2000 < 0$$

which corresponds to a local maximum.

For maximum revenue, the monthly fee should be \$3, and this gives a total revenue of \$9000.

Example 17

A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x, of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution

Number of production runs per year $= \dfrac{10\,000}{x}$

Set-up costs for these production runs $= 40\left(\dfrac{10\,000}{x}\right)$

Let C be the total set-up and carrying costs. Then

$$C = x^{\frac{3}{2}} + \frac{400\,000}{x}$$

$$= x^{\frac{3}{2}} + 400\,000x^{-1}, \quad x > 0$$

$$\therefore \quad \frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{400\,000}{x^2}$$

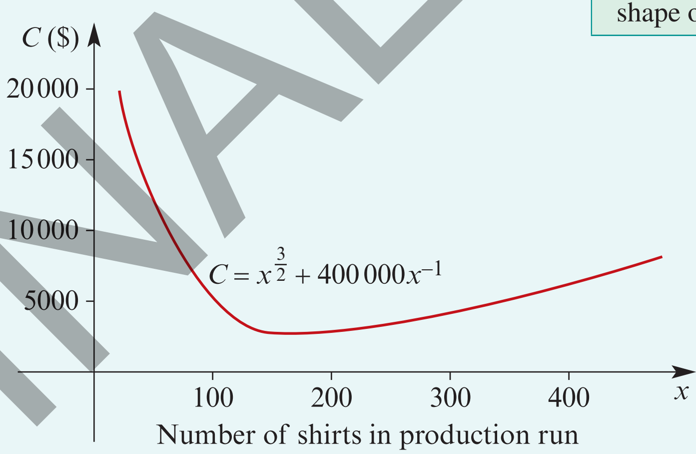
Thus $\dfrac{dC}{dx} = 0$ implies $\dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{400\,000}{x^2}$

$$x^{\frac{5}{2}} = \frac{400\,000 \times 2}{3}$$

$$\therefore \quad x \approx 148.04$$

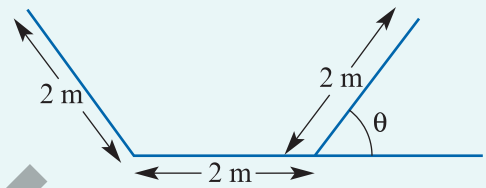
Each production run should be 148 shirts.

x		148.04	
$\frac{dC}{dx}$	−	0	+
shape of C	\	—	/



Example 18

The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution

Let A m^2 be the area of the trapezium. Then

$$A = \frac{1}{2} \times 2\sin\theta \times (2 + 2 + 4 \times \cos\theta)$$
$$= \sin\theta \cdot (4 + 4\cos\theta)$$

and $A'(\theta) = \cos\theta \cdot (4 + 4\cos\theta) - 4\sin^2\theta$

$$= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta)$$
$$= 4\cos\theta + 8\cos^2\theta - 4$$

The maximum will occur when $A'(\theta) = 0$:

$$8\cos^2\theta + 4\cos\theta - 4 = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

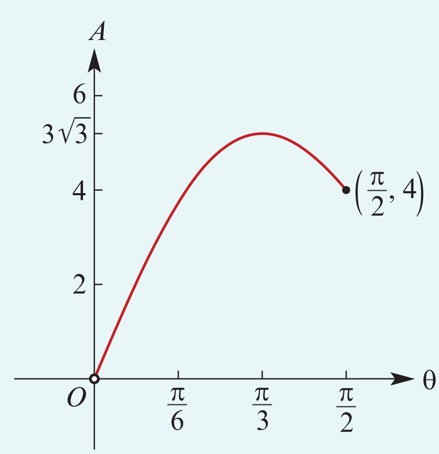
The practical restriction on θ is that $0 < \theta \le \frac{\pi}{2}$.

Therefore the only possible solution is $\theta = \frac{\pi}{3}$, and a gradient chart confirms that $\frac{\pi}{3}$ gives a maximum.

θ		$\frac{\pi}{3}$	
$A'(\theta)$	+	0	−
shape of A	/	—	\

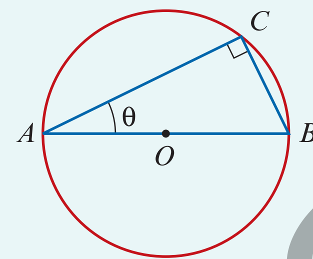
When $\theta = \frac{\pi}{3}$, $A = \frac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$,

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.



Example 19

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0° < \theta° < 90°)$.

Solution

a Time taken $= \dfrac{\text{distance travelled}}{\text{speed}}$

Therefore the swim takes $\frac{4\cos\theta}{3}$ hours and the walk takes $\frac{4\theta}{4}$ hours.

Thus the total time taken is given by $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

b $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\frac{dT}{d\theta} = 0$, and $\frac{1}{3}(-4\sin\theta + 3) = 0$ implies $\sin\theta = \frac{3}{4}$.

Therefore $\theta = 48.59°$ to two decimal places.

Second derivative test:

Note that $\frac{d^2T}{d\theta^2} = -\frac{4}{3}\cos\theta$. When $\theta = 48.59°$, we have $\cos\theta > 0$ and so $\frac{d^2T}{d\theta^2} < 0$.

Hence the value of T is a maximum when $\theta = 48.59°$.

Note: The maximum time taken is 1.73 hours. If the man swims straight across the lake, it takes $1\frac{1}{3}$ hours. If he walks all the way, it takes approximately 1.57 hours.

Example 20

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution

$$N(t) = 36te^{-0.1t}$$

$$\therefore \quad N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$$

$$= e^{-0.1t}(36 - 3.6t)$$

Thus $N'(t) = 0$ implies $t = 10$.

The maximum population is $N(10) = 360e^{-1} \approx 132$.

Maximum rates of increase and decrease

We know that we can use the derivative of a function to help find the maximum and minimum values of the function. Similarly, we can use the second derivative to help find the maximum rate of increase or decrease of the function.

The second derivative gives the instantaneous rate of change of the derivative.

- If $\dfrac{d^2y}{dx^2} > 0$, then $\dfrac{dy}{dx}$ is increasing as x increases.
- If $\dfrac{d^2y}{dx^2} < 0$, then $\dfrac{dy}{dx}$ is decreasing as x increases.

Example 21

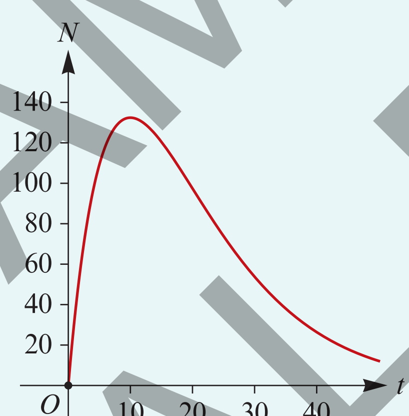
Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

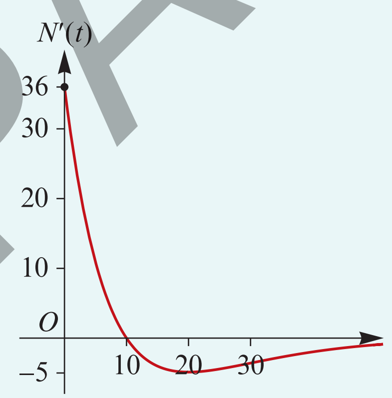
b Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution

a $N(t) = 36te^{-0.1t}$



$N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$



b The rate of change of the population is $N'(t) = e^{-0.1t}(36 - 3.6t)$.
From the graph, the maximum value of $N'(t)$ occurs at $t = 0$. Thus the maximum rate of increase of the population is $N'(0) = 36$ bacteria per unit of time.

We now calculate

$$N''(t) = -7.2e^{-0.1t} + 0.36te^{-0.1t}$$
$$= e^{-0.1t}(-7.2 + 0.36t)$$

Thus $N''(t) = 0$ implies $t = 20$.

The minimum value of $N'(t)$ occurs at $t = 20$. Since $N'(20) = -36e^{-2} \approx -4.9$, the maximum rate of decrease of the population is 4.9 bacteria per unit of time.

Note: The graph of the original function N has a point of inflection at $t = 20$.

Section summary

Here are some steps for solving optimisation problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Exercise 12D

SF

Example 13 **1** Find the maximum area of a rectangular field that can be enclosed by 100 m of fencing.

Example 14 **2** Find two positive numbers that sum to 4 and such that the sum of the cube of the first and the square of the second is as small as possible.

3 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P. (Use the second derivative test.)

4 A farmer has 4 km of fencing wire and wishes to fence a rectangular piece of land through which flows a straight river, which is to be utilised as one side of the enclosure. How can this be done to enclose as much land as possible?

5 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum. (Use the second derivative test.)

CF

Example 15 **6** A cuboid has a total surface area of 150 cm^2 with a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

Example 16, 17 **7** A manufacturer finds that the daily profit, \$$P$, from selling n articles is given by $P = 100n - 0.4n^2 - 160$.

a **i** Find the value of n which maximises the daily profit.

ii Find the maximum daily profit.

b Sketch the graph of P against n. (Use a continuous graph.)

c State the allowable values of n for a profit to be made.

d Find the value of n which maximises the profit per article.

CF

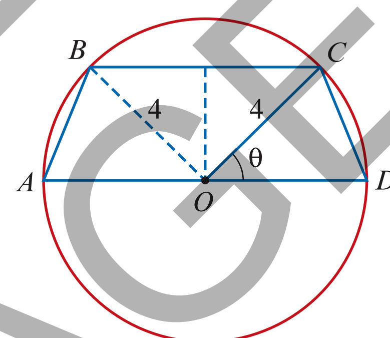
8 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$ with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that produces the maximum number of salmon swimming upstream.

9 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos.

Example 18 **10** $ABCD$ is a trapezium with $AB = CD$. The vertices are on a circle with centre O and radius 4 units. The line segment AD is a diameter of the circle.



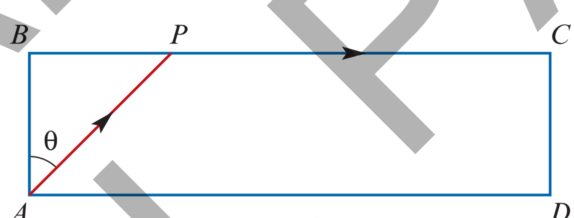
a Find BC in terms of θ.

b Find the area of the trapezium in terms of θ and hence find the maximum area.

11 Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

CU

Example 19 **12** The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



$AB = 300$ m

$BC = 1100$ m

a An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.

b The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.

c Use the results from **a** and **b** to show that the total time taken, T seconds, is given by $T = 220 + \dfrac{75 - 60\sin\theta}{\cos\theta}$.

d Find $\dfrac{dT}{d\theta}$.

e Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.

f Find the minimum value of T and find the distance of point P from B that will minimise the athlete's running time.

Example 20 **13** The number, $N(t)$, of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Example 21 **14** The number, $N(t)$, of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

b Find the maximum rates of increase and decrease of the population and the times at which these occur.

15 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by

$$V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right), \quad 0 \le t \le 20$$

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find $V'(t)$, the rate of flow of water into the flask.

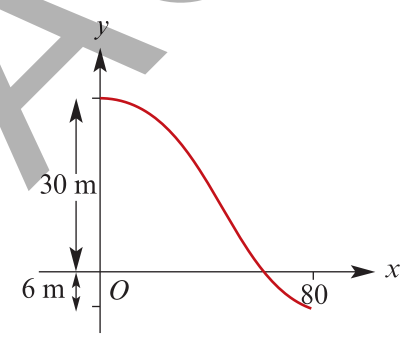
c Sketch the graph of $V(t)$ against t for $0 \le t \le 20$.

d Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

e At what time is the flow greatest and what is the flow at this time?

16 A section of a roller coaster can be described by the rule

$$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, \quad 0 \le x \le 80$$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

17 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

Chapter summary

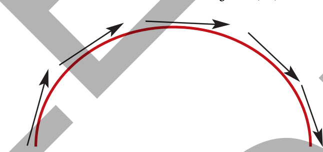
The second derivative

- The **second derivative** of a function f is the derivative of the derivative of f, and it is denoted by f''.
- In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Using the second derivative in graph sketching

- Concave up: $f''(x) > 0$

- Concave down: $f''(x) < 0$



- A **point of inflection** is where the curve changes from concave up to concave down or from concave down to concave up.
- At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.
- Second derivative test

  For the graph of $y = f(x)$:
  - If $f'(a) = 0$ and $f''(a) > 0$, then the point $(a, f(a))$ is a local minimum.
  - If $f'(a) = 0$ and $f''(a) < 0$, then the point $(a, f(a))$ is a local maximum.
  - If $f''(a) = 0$, then further investigation is necessary.

Maximum and minimum values

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \le M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \ge N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Motion in a straight line

For an object moving in a straight line with position x at time t:

- velocity $v = \dfrac{dx}{dt}$
- acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

Technology-free questions

1 Find the second derivative of each of the following:

a $(2x + 5)^8$ **b** $\sin(2x)$ **c** $\cos\left(\dfrac{x}{3}\right)$

d e^{-4x} **e** $\ln(6x)$ **f** $\ln(\sin x)$

SF

Review

SF

2 For each of the following curves, state the coordinates of the point(s) of inflection:

a $y = x^3 - 8x^2$ **b** $y = \sin\left(x - \frac{\pi}{6}\right)$, $0 \le x \le 2\pi$

c $y = \ln x + \frac{1}{x}$ **d** $y = \frac{x}{\ln x}$

3 Consider the graph of $f(x) = x^2 \ln x$.

a Find the value of x for which there is a local minimum.

b Find the value of x for which there is a point of inflection.

4 Consider the graph of $f(x) = x^3 e^x$.

a Find the value of x for which there is a local minimum.

b Find the value of x for which there is a stationary point of inflection.

CF

5 Consider the graph of $f(x) = ce^{-kx^2}$, where c and k are constants.

a Find $f'(x)$ and $f''(x)$ in terms of c and k.

b Find the coordinates of the points of inflection if $c = 1$ and $k = 2$.

c Find the value of k for which there are points of inflection at $x = \frac{1}{4}$ and $x = -\frac{1}{4}$.

6 A particle moves in a straight line. Its position, x m, at time t seconds is given by

$$x = \frac{1}{9}\left(1 - (3t + 1)e^{-3t}\right)$$

a Find the velocity of the particle at time t.

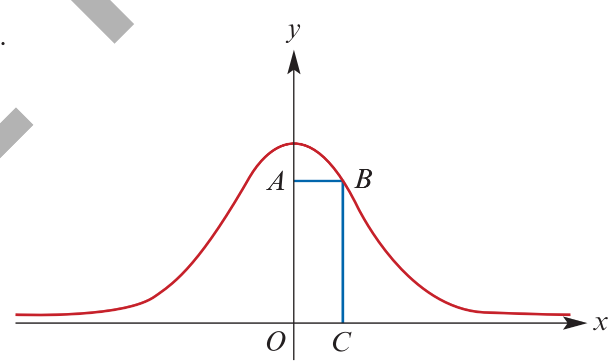
b Find the acceleration of the particle at time t.

c Find the maximum velocity of the particle and the time and location at which this maximum velocity occurs.

7 The graph of $f(x) = e^{-x^2}$ is shown.

Let $B(x, f(x))$ be a point on the graph, where $x > 0$.

Draw the rectangle $OABC$ as shown, where A is on the y-axis and C is on the x-axis.



a Find an expression for the area, W, of the rectangle in terms of x.

b Find $\frac{dW}{dx}$ and $\frac{d^2W}{dx^2}$.

c Find the coordinates of the point of inflection on the graph of W against x.

d Find the maximum value of W and the value of x for which this occurs. Verify that the point is a local maximum by showing that $\frac{d^2W}{dx^2} < 0$.

Review

Multiple-choice questions

1 For a polynomial function with rule $f(x)$, the derivative satisfies $f'(a) = f'(b) = 0$, $f'(x) > 0$ for $x \in (a, b)$, $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > b$. The nature of the stationary points of the graph of $y = f(x)$ is

- **A** local maximum at $(a, f(a))$ and local minimum at $(b, f(b))$
- **B** local minimum at $(a, f(a))$ and local maximum at $(b, f(b))$
- **C** stationary point of inflection at $(a, f(a))$ and local minimum at $(b, f(b))$
- **D** stationary point of inflection at $(a, f(a))$ and local maximum at $(b, f(b))$
- **E** local minimum at $(a, f(a))$ and stationary point of inflection at $(b, f(b))$

2 The second derivative of e^{-x^3} is

A e^{-6x} **B** $-3x^2e^{-x^3}$ **C** $-6x^4e^{-x^3}$
D $3x(3x^3 - 2)e^{-x^3}$ **E** $(9x^4 - 2)e^{-x^3}$

3 The graph of $y = 5x^4 - x^5$ has a point of inflection at

A $(0, 0)$ only **B** $(3, 162)$ only **C** $(4, 256)$ only
D $(0, 0)$ and $(3, 162)$ **E** $(0, 0)$ and $(4, 256)$

4 The position of a particle moving along the x-axis is given by $x(t) = \sin(2t) - \cos(3t)$ at time $t \geq 0$. When $t = \pi$, the acceleration of the particle is

A 9 **B** 19 **C** 0 **D** −19 **E** −9

5 The volume, V cm^3, of a solid is given by the formula $V = -10x(2x^2 - 6)$, where x cm is a particular measurement. The value of x for which the volume is a maximum is

A 0 **B** 1 **C** $\sqrt{2}$ **D** $\sqrt{3}$ **E** 2

6 If $f(x) = ax^4 + bx^2$, where $a > 0$ and $b > 0$, then which of the following must be true?

A There are no stationary points. **B** There are two stationary points.
C The graph is concave up for all x. **D** The graph is concave down for all x.
E There is one point of inflection.

7 The coordinates of the points of inflection of $y = \sin x$ for $x \in [0, 2\pi]$ are

A $\left(\frac{\pi}{2}, 1\right)$ and $\left(-\frac{\pi}{2}, -1\right)$ **B** $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$
C $(\pi, 0)$ **D** $(1, 0)$ **E** $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$

8 Let $g(x) = e^{f(x)}$, where the function f is twice differentiable. Then $g''(x)$ is equal to

A $f'(x)e^{f(x)}$ **B** $f''(x)e^{f(x)} + (f'(x))^2e^{f(x)}$
C $e^{f''(x)}$ **D** $f''(x)e^{f(x)}$ **E** $f''(x)e^{f(x)} + f'(x)e^{f(x)}$

Review

9 Let $g(x) = e^{-x} f(x)$, where the function f is twice differentiable. There is a point of inflection on the graph of $y = g(x)$ at $(a, g(a))$. An expression for $f''(a)$ in terms of $f'(a)$ and $f(a)$ is

A $f''(a) = f(a) + f'(a)$ **B** $f''(a) = 2f(a)f'(a)$ **C** $f''(a) = 2f(a) + f'(a)$

D $f''(a) = \dfrac{f'(a)}{f(a)}$ **E** $f''(a) = 2f'(a) - f(a)$

10 The graph of a polynomial function with rule $y = f(x)$ has a local maximum at the point with coordinates $(a, f(a))$. The graph also has a local minimum at the origin, but no other stationary points. The graph of the function with rule $y = -2f\left(\dfrac{x}{2}\right) + k$, where k is a positive real number, has

A a local minimum at the point with coordinates $(2a, -2f(a) + k)$

B a local maximum at the point with coordinates $(2a, -2f(a) + k)$

C a local minimum at the point with coordinates $\left(\dfrac{a}{2}, 2f(a) + k\right)$

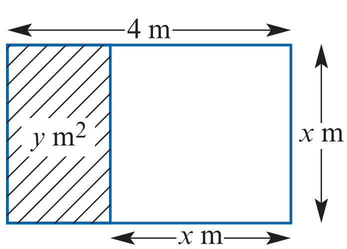
D a local maximum at the point with coordinates $\left(\dfrac{a}{2}, -2f(a) + k\right)$

E a local maximum at the point with coordinates $(2a, -2f(a) - k)$

Extended-response questions

CF

1 The diagram shows a rectangle with sides 4 m and x m and a square with side x m. The area of the shaded region is y m^2.



a Find an expression for y in terms of x.

b Find the possible values for x.

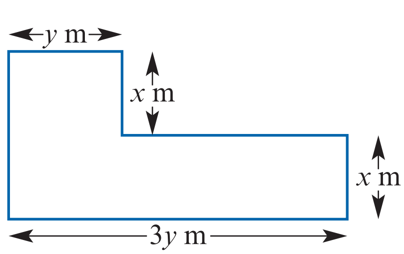
c Find the maximum value of y and the corresponding value of x.

d Explain briefly why this value of y is a maximum.

e Sketch the graph of y against x.

f State the possible values for y.

2 A flower bed is to be L-shaped, as shown in the figure, and its perimeter is 48 m.



a Write down an expression for the area, A m^2, in terms of y and x.

b Find y in terms of x.

c Write down an expression for A in terms of x.

d Find the values of x and y that give the maximum area.

e Find the maximum area.

3 It costs $(12 + 0.008x)$ dollars per kilometre to operate a truck at x km/h. In addition it costs \$14.40 per hour to pay the driver.

a What is the total cost per kilometre if the truck is driven at:

i 40 km/h

ii 64 km/h?

b Write an expression for C, the total cost per kilometre, in terms of x.

c Sketch the graph of C against x for $0 < x < 120$.

d At what speed should the truck be driven to minimise the total cost per kilometre?

4 A box is to be made from a 10 cm by 16 cm sheet of metal by cutting equal squares out of the corners and bending up the flaps to form the box. Let the lengths of the sides of the squares be x cm and let the volume of the box formed be V cm^3.

a Show that $V = 4(x^3 - 13x^2 + 40x)$.

b State the set of x-values for which the expression for V in terms of x is valid.

c Find the values of x such that $\frac{dV}{dx} = 0$.

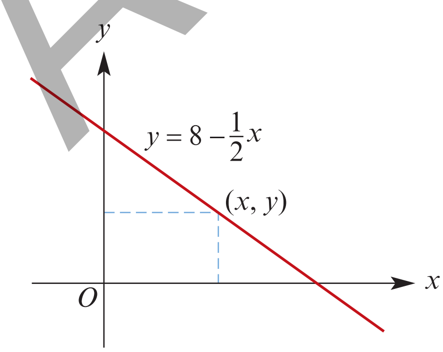
d Find the dimensions of the box if the volume is to be a maximum.

e Find the maximum volume of the box.

f Sketch the graph of V against x for the domain established in **b**.

5 A rectangle has one vertex at the origin, another on the positive x-axis, another on the positive y-axis and a fourth on the line $y = 8 - \frac{x}{2}$.

What is the greatest area the rectangle can have?



6 At a factory the time, T seconds, spent in producing a certain size metal component is related to its weight, w kg, by $T = k + 2w^2$, where k is a constant.

a If a 5 kg component takes 75 seconds to produce, find k.

b Sketch the graph of T against w.

c Write down an expression for the average time A (in seconds per kilogram).

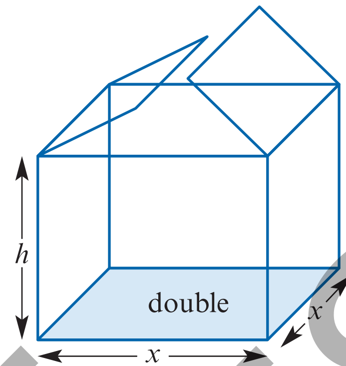
d **i** Find the weight that yields the minimum average machining time per kilogram.

ii State the minimum average machining time.

7 An open tank is to be constructed with a square base and vertical sides to contain 500 m^3 of water. What must be the area of sheet metal used in its construction if this area is to be a minimum?

Review

8 A manufacturer produces cardboard boxes that have a square base. The top of each box consists of a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. Each box must have a volume of 12 cubic metres.



a Show that the area of cardboard required is given by $C = 3x^2 + 4xh$.

b Express C as a function of x only.

c Sketch the graph of C against x for $x > 0$.

d **i** What dimensions of the box will minimise the amount of cardboard used?

ii What is the minimum area of cardboard used?

9 A piece of wire of length 1 m is bent into the shape of a sector of a circle of radius a cm and sector angle θ. Let the area of the sector be A cm^2.

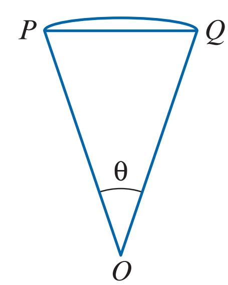
a Find A in terms of a and θ.

b Find A in terms of θ.

c Find the value of θ for which A is a maximum.

d Find the maximum area of the sector.

10 A piece of wire of fixed length, L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



a Show that the area, A cm^2, of the sector is given by

$$A = \frac{1}{2}rL - r^2$$

b **i** Find a relationship between r and L for which $\frac{dA}{dr} = 0$.

ii Find the corresponding value of θ.

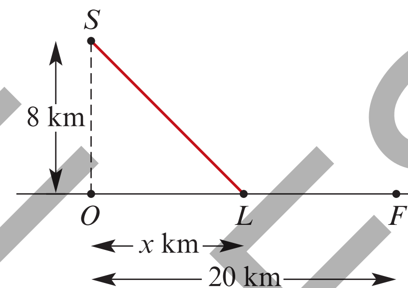
iii Determine the nature of the stationary point found in **i**.

c Show that, for the value of θ found in **b ii**, the area of the triangle OPQ is approximately 45.5% of the area of sector OPQ.

11 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

12 At noon the captain of a ship sees two fishing boats approaching. One of them is 10 km due east and travelling west at 8 km/h. The other is 6 km due north and travelling south at 6 km/h. At what time will the fishing boats be closest together and how far apart will they be?

13 The point S is 8 km offshore from the point O, which is located on the straight shore of a lake, as shown in the diagram. The point F is on the shore, 20 km from O. Contestants race from the start, S, to the finish, F, by rowing in a straight line to some point, L, on the shore and then running along the shore to F. A certain contestant rows at 5 km per hour and runs at 15 km per hour.

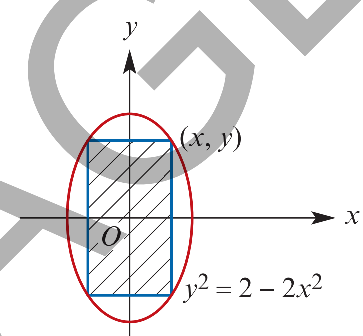


a Show that, if the distance OL is x km, the time taken by this contestant to complete the course is (in hours):

$$T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

b Show that the time taken by this contestant to complete the course has its minimum value when $x = 2\sqrt{2}$. Find this time.

14 A rectangular beam is to be cut from a non-circular tree trunk whose cross-sectional outline can be represented by the equation $y^2 = 2 - 2x^2$.



a Show that the area of the cross-section of the beam is given by

$$A = 4x\sqrt{2 - 2x^2}$$

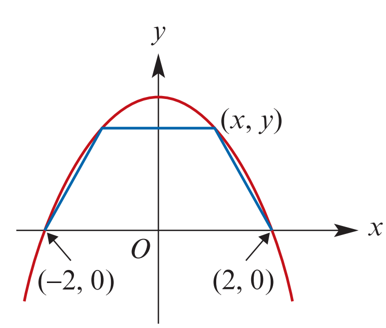
where x is the half-width of the beam.

b State the possible values for x.

c Find the value of x for which the cross-sectional area of the beam is a maximum and find the corresponding value of y.

d Find the maximum cross-sectional area of the beam.

15 An isosceles trapezium is inscribed in the parabola $y = 4 - x^2$ as illustrated.



a Show that the area of the trapezium is given by $\frac{1}{2}(4 - x^2)(2x + 4)$.

b Show that the trapezium has its greatest area when $x = \frac{2}{3}$.

c Repeat with the parabola $y = a^2 - x^2$:

i Show that the area, A, of the trapezium is given by $(a^2 - x^2)(a + x)$.

ii Use the product rule to find $\frac{dA}{dx}$.

iii Show that a maximum occurs when $x = \frac{a}{3}$.

16 Consider the function with rule $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$. CU

a **i** If $x + 1$ is a factor of $f(x)$, find the value of a.

ii Using a calculator, plot the graph of $y = f(x)$ for this value of a.

b Let $g(x) = 6x^4 - x^3 + 21x^2 - 6x + 8$.

i Plot the graph of $y = g(x)$.

ii Find the minimum value of $g(x)$ and the value of x for which this occurs.

iii Find $g'(x)$.

iv Using a calculator, solve the equation $g'(x) = 0$ for x.

v Find $g'(0)$ and $g'(10)$.

vi Find $g''(x)$.

vii Show that the graph of $y = g'(x)$ has no stationary points and thus deduce that $g'(x) = 0$ has only one solution.

17 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by $f(x) = x \ln x + 1$ for $x \in (0, 6]$, where $f(x)$ is the ability to memorise at age x months.

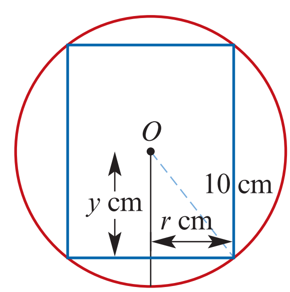
a Find $f'(x)$.

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

18 A cylinder is to be cut from a sphere. The cross-section through the centre of the sphere is as shown. The radius of the sphere is 10 cm. Let r cm be the radius of the cylinder.



a **i** Find y in terms of r and hence the height, h cm, of the cylinder.

ii The volume of a cylinder is given by $V = \pi r^2 h$. Find V in terms of r.

b **i** Plot the graph of V against r using a calculator.

ii Find the maximum volume of the cylinder and the corresponding values of r and h. (Use a calculator.)

iii Find the two possible values of r if the volume is 2000 cm^3.

c **i** Find $\frac{dV}{dr}$.

ii Hence find the exact value of the maximum volume and the value of r for which this occurs.

d **i** Plot the graph of the derivative function $\frac{dV}{dr}$ against r, using a calculator.

ii From the calculator, find the values of r for which $\frac{dV}{dr}$ is positive.

iii From the calculator, find the values of r for which $\frac{dV}{dr}$ is increasing.

19 Consider the curve with equation $y = (x^2 - 2x)e^x$.

a Find the x-axis intercepts.

b Find the equation of the tangent to the curve at $x = 1$.

c Find the equation of the tangent to the curve at $x = 2$.

d Find the x-values for which there is a turning point.

e Find the x-values for which there is a point of inflection.

f Sketch the curve.

CF

20 Consider the family of curves with equations $y = (2x^2 - 5x)e^{ax}$, where $a \in \mathbb{R} \setminus \{0\}$.

CU

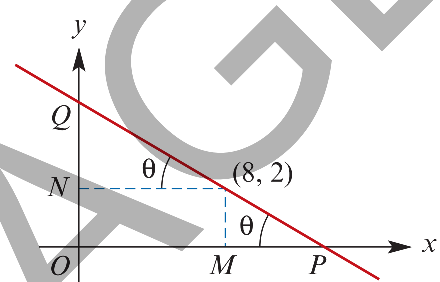
a Find the x-axis intercepts.

b Find the x-values for which there is a turning point, in terms of a.

c Find the x-values for which there is a point of inflection, in terms of a.

d If a curve from this family passes through the point (3, 10), find the value of a.

21 A straight line is drawn through the point (8, 2) to intersect the positive y-axis at Q and the positive x-axis at P. (In this question, we find the minimum value of $OP + OQ$.)



a Show that $\dfrac{d}{d\theta}\left(\dfrac{1}{\tan\theta}\right) = -\dfrac{1}{\sin^2\theta}$.

b Find MP in terms of θ.

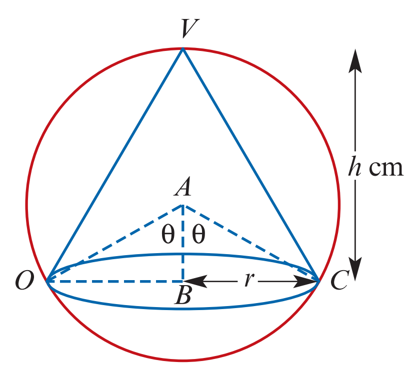
c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\dfrac{dx}{d\theta}$.

f Find the minimum value of x and the value of θ for which this occurs.

22 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and both the angles $\angle OAB$ and $\angle CAB$ have magnitude θ. The height of the cone is h cm and the radius of the cone is r cm.



a Find h in terms of a and θ.

b Find r in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \dfrac{1}{3}\pi r^2 h$.

c Use the results from **a** and **b** to show that

$$V = \frac{1}{3}\pi a^3 \sin^2\theta \cdot (1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (where a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

Review

23 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where

$$y = \frac{Ae^{bt}}{1 + Ae^{bt}} \qquad (1)$$

and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\frac{dy}{dt}$ in terms of t.

c From equation (1), show that $Ae^{bt} = \frac{y}{1 - y}$.

d **i** Show that $\frac{dy}{dt} = by(1 - y)$.

ii Hence, or otherwise, show that the maximum value of $\frac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$, find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

24 Let $f(x) = \frac{e^x}{x}$ for $x > 0$.

a Find $f'(x)$.

b Find the value of x such that $f'(x) = 0$.

c Find the coordinates of the stationary point and state its nature.

d **i** Find $\frac{f'(x)}{f(x)}$. **ii** Find $\lim_{x \to \infty} \frac{f'(x)}{f(x)}$ and comment.

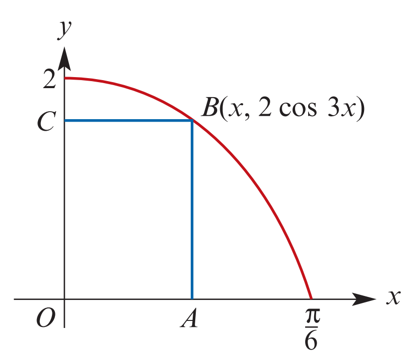
e Sketch the graph of f.

f Over a period of years, the number of birds (n) in an island colony decreased and increased with time (t years) according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

25 A section of the graph of $y = 2\cos(3x)$ is shown in the diagram.



a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos(3x)$.

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator, plot the graph of $A = 2x\cos(3x)$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.