7 Refresher on differentiation

Objectives

- ▶ To understand the concept of **limit**.
- ▶ To understand the definition of **differentiation**.
- ▶ To understand and use the notation for the **derivative** of a function.
- ▶ To find the **gradient** of a tangent to the graph of a polynomial function by calculating its derivative.
- ▶ To deduce the **graph of the derivative** from the graph of a function.

Throughout this course, we have been using different types of functions as mathematical models of the relationship between two variables. We have used the idea that one variable, say y, is a function of another variable, say x.

When establishing and applying such a mathematical model, it is important to understand how the relationship between the two variables is changing. For example, if x increases, does y also increase, or does it decrease, or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely, etc.?

We can study rates of change using differential calculus. It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

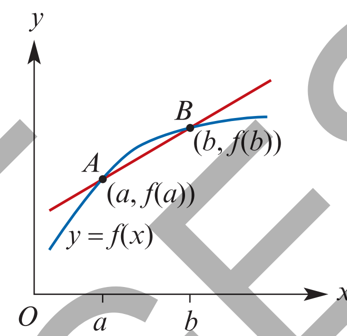
In this chapter, we revise some of the important ideas and results from your study of differential calculus in Mathematical Methods Units 1 & 2.

7A The derivative

We begin this chapter by recalling the definition of average rate of change from Mathematical Methods Units 1 & 2.

▶ Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through the two points $A(a, f(a))$ and $B(b, f(b))$.



That is:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

Example 1

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

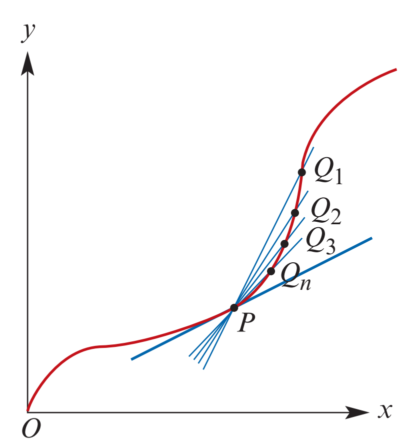
$$\text{Average rate of change} = \frac{20 - 4}{5 - 1} = 4$$

▶ The tangent to a curve at a point

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

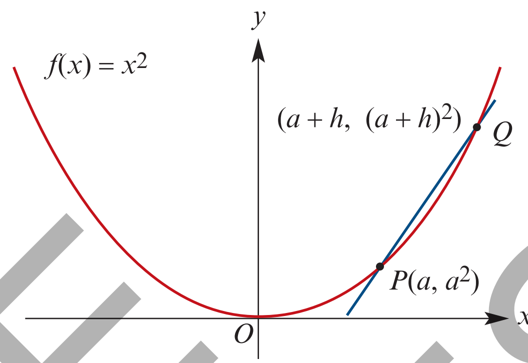
The **instantaneous rate of change** at P can be defined by considering what happens when we look at a sequence of secants $PQ_1, PQ_2, PQ_3, \ldots, PQ_n, \ldots$, where the points Q_i get closer and closer to P.

Here we first focus our attention on the gradient of the tangent at P.



Consider the function $f(x) = x^2$.

The gradient of the secant PQ shown on the graph is



$$\text{gradient of } PQ = \frac{(a+h)^2 - a^2}{a + h - a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{h}$$
$$= 2a + h$$

The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

Note: This also can be interpreted as the instantaneous rate of change of f at $(a, f(a))$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

Limit notation

The notation for the limit of $2x + h$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 2

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\text{Gradient of } PQ = \frac{(2+h)^3 - 8}{2 + h - 2}$$
$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= \frac{12h + 6h^2 + h^3}{h}$$
$$= 12 + 6h + h^2$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

The following example provides practice in determining limits.

Example 3

Find:

a $\lim_{h \to 0} (22x^2 + 20xh + h)$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h \to 0} 12x$

d $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} (22x^2 + 20xh + h) = 22x^2$

b $$\begin{aligned} \lim_{h \to 0} \frac{3x^2h + 2h^2}{h} &= \lim_{h \to 0} (3x^2 + 2h) \\ &= 3x^2 \end{aligned}$$

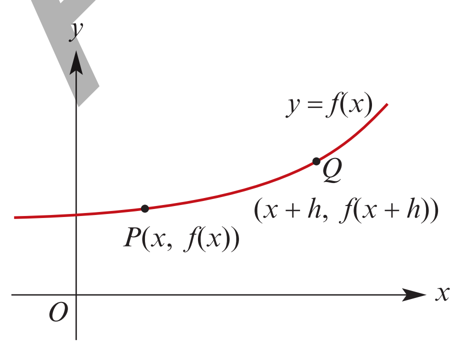
c $\lim_{h \to 0} 12x = 12x$

d $\lim_{h \to 0} 4 = 4$

Definition of the derivative

In general, consider the graph of $y = f(x)$, where f is a function.

$$\begin{aligned} \text{Gradient of secant } PQ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.

Derivative of a function

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Section 7D.

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 4

Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$

b $f(x) = 2 - x^3$

Solution

a
$$\frac{f(x+h) - f(x)}{h} = \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h}$$
$$= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h}$$
$$= \frac{6xh + 3h^2 + 2h}{h}$$
$$= 6x + 3h + 2$$

Therefore
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

b
$$\frac{f(x+h) - f(x)}{h} = \frac{2 - (x+h)^3 - (2 - x^3)}{h}$$
$$= \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - 2 + x^3}{h}$$
$$= \frac{-3x^2h - 3xh^2 - h^3}{h}$$
$$= -3x^2 - 3xh - h^2$$

Therefore
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-3x^2 - 3xh - h^2) = -3x^2$$

Section summary

- The **derivative** of the function f is denoted f' and is defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Exercise 7A

SF

Example 1

1 Find the average rate of change of the function with rule $f(x) = -x^2 + 2x + 1$ as x changes from -1 to 4.

2 Find the average rate of change of the function with rule $f(x) = 6 - x^3$ as x changes from -1 to 1.

Example 2

3 For the curve with equation $y = x^2 + 5x$:

a Find the gradient of the secant through points P and Q, where P is the point $(2, 14)$ and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(2, 14)$.

Example 3

4 Find:

a $\lim_{h\to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim_{h\to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim_{h\to 0} (40 - 50h)$

d $\lim_{h\to 0} 5h$

e $\lim_{h\to 0} 5$

f $\lim_{h\to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim_{h\to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim_{h\to 0} 3x$

i $\lim_{h\to 0} \dfrac{3x^3h - 5x^2h^2 + xh}{h}$

j $\lim_{h\to 0} (6x - 7h)$

5 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ, where P is the point $(1, 0)$ and Q is the point $(1 + h, (1 + h)^3 - (1 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(1, 0)$.

6 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x + h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

7 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h, and hence find the gradient of the tangent to the curve $y = x^2 + 2x + 5$ at $x = 2$.

Example 4

8 For each of the following, find $f'(x)$ by finding $\lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$:

a $f(x) = 5x^2$

b $f(x) = 3x + 2$

c $f(x) = 5$

d $f(x) = 3x^2 + 4x + 3$

e $f(x) = 5x^3 - 5$

f $f(x) = 5x^2 - 6x$

7B Rules for differentiation

The derivative of x^n where n is a positive integer

Differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result:

> For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

Proof We can prove this result using the binomial theorem (discussed in Appendix A).

Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1x^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + {}^nC_{n-1}xh^{n-1} + h^n - x^n \\ &= {}^nC_1x^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + {}^nC_{n-1}xh^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + {}^nC_{n-1}xh^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2x^{n-2}h^2 + \cdots + {}^nC_{n-1}xh^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2x^{n-2}h + \cdots + {}^nC_{n-1}xh^{n-2} + h^{n-1} \end{aligned}$$

Thus
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2x^{n-2}h + \cdots + {}^nC_{n-1}xh^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

We will revise the rules for the derivatives of products and quotients in Chapter 8.

The process of finding the derivative function is called **differentiation**.

Example 5

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $\quad f(x) = x^5 - 2x^3 + 2$

Then $\quad f'(x) = 5x^4 - 2(3x^2) + 0$

$\qquad\qquad = 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 6

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $\quad f(x) = 3x^3 - 6x^2 + 1$

Then $\quad f'(x) = 3(3x^2) - 6(2x) + 0$

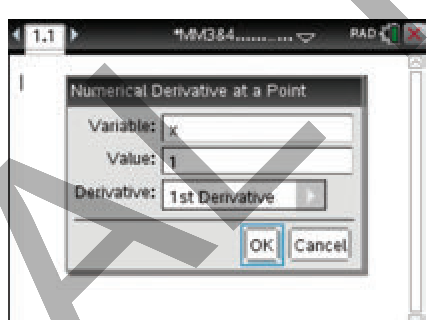
$\qquad\qquad = 9x^2 - 12x$

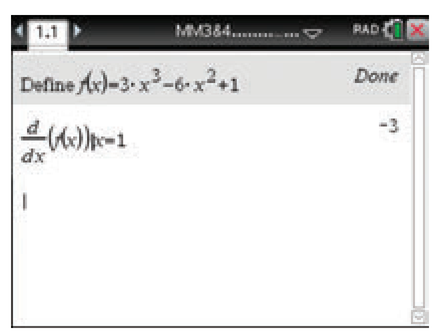
$\therefore \quad f'(1) = 9 - 12 = -3$

TI-84

Using the TI-Nspire CX non-CAS

- In a **Calculator** application, define $f(x) = 3x^3 - 6x^2 + 1$.
- To find the value of the derivative at $x = 1$, use menu > **Calculus** > **Numerical Derivative at a Point** and complete as shown.





Note: The derivative template can also be accessed from the 2D-template palette or by using shift –.

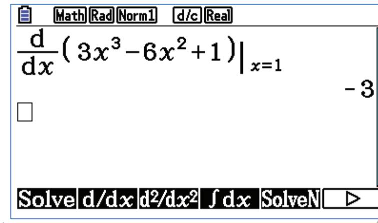
Using the Casio

To find the derivative of $3x^3 - 6x^2 + 1$ at $x = 1$:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to **Calculation** OPTN F4, then **d/dx** F2.
- Enter the expression $3x^3 - 6x^2 + 1$ and the x-value 1:

  3 X,θ,T ∧ 3 ► − 6 X,θ,T x^2 + 1

  ► 1 EXE



Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 7A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 7

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\dfrac{dy}{dx}$, and so we write $\dfrac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\dfrac{dy}{dx}$.

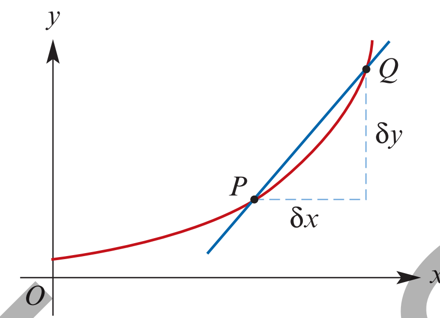
Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\dfrac{dz}{dt}$.

Warning: In Leibniz notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



Example 8

a If $y = t^2$, find $\frac{dy}{dt}$.

b If $x = t^3 + t$, find $\frac{dx}{dt}$.

c If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a $y = t^2$

$\frac{dy}{dt} = 2t$

b $x = t^3 + t$

$\frac{dx}{dt} = 3t^2 + 1$

c $z = \frac{1}{3}x^3 + x^2$

$\frac{dz}{dx} = x^2 + 2x$

Example 9

a For $y = (x + 3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t - 1)^2(t + 2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First write $y = (x + 3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$

$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$

$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$

$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d

$$y = 2x^3 - 1$$

$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

Example 10

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3) = 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z) = 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2) = 18z^2 - 8z$

Example 11

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8

b $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = -6

Solution

a $y = x^3$ implies $\frac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$

The points are $\left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right)$ and $\left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\frac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\frac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

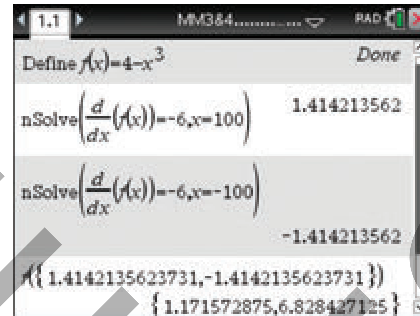
The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

TI-84

Using the TI-Nspire CX non-CAS

The calculator can be used to obtain an approximate solution for Example 11c.

- Define $f(x) = 4 - x^3$.
- Solve the equation $\frac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.
- The points are (1.41, 1.17) and (−1.41, 6.83), correct to two decimal places.



Note: Check the number of solutions by finding the intersection points of the graphs of $f1(x) = \frac{d}{dx}(f(x))$ and $f2(x) = -6$.

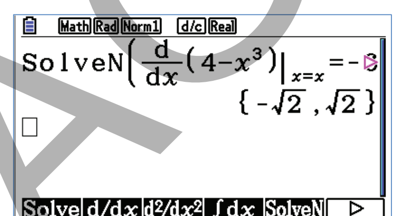
Using the Casio

Method 1: Using the numerical solver

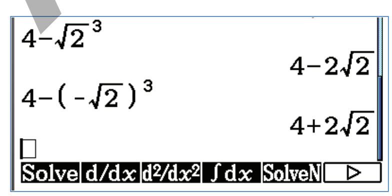
For Example 11c:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** (OPTN) (F4), then **SolveN** (F5).
- Enter the equation $\frac{d}{dx}(4 - x^3) = -6$ as follows:

  (F2) (4) (−) (X,θ,T) (∧) (3) ▶ ▶ (X,θ,T) ▶
  (SHIFT) (•) ((−)) (6) ( )) (EXE)



- Substitute each x-value into the rule $y = 4 - x^3$ to find the corresponding y-value.
- The points are $(\sqrt{2}, 4 - 2\sqrt{2})$ and $(-\sqrt{2}, 4 + 2\sqrt{2})$.

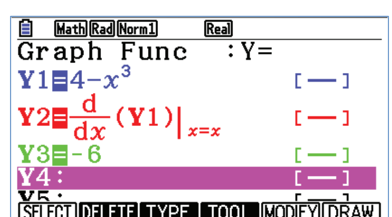


Method 2: Using Graph mode

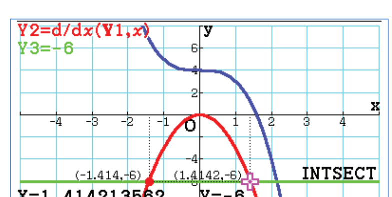
For Example 11c:

- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = 4 - x^3$ in $Y1$.
- Enter the derivative of $Y1$ in $Y2$ as follows:

  (OPTN) (F2) (F1) (F1) (1) ▶ (X,θ,T) (EXE)



- Enter the rule $y = -6$ in $Y3$.
- Select **Draw** (F6) to view the graphs.
- Adjust the View Window if required.
- Go to **G-Solve** (SHIFT) (F5), then **Intersection** (F5). Select the graph of $Y2$ and the graph of $Y3$.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example, if $\alpha = 135°$, then $\tan\alpha = -1$ and so the gradient is -1.

Example 12

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$2x - 7 = 1 \quad (\text{as } \tan 45° = 1)$

$2x = 8$

$\therefore \quad x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore \quad x = \dfrac{5}{2}$

The coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$.

Increasing and decreasing functions

We have discussed strictly increasing and strictly decreasing functions in Chapter 1.

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

We have the following very important results.

If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

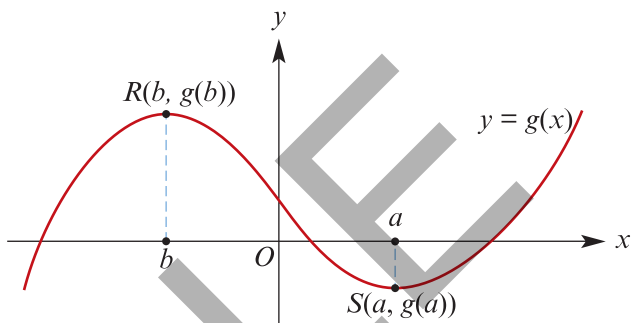
If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f(x) = x^3$, $x \in \mathbb{R}$, is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

Sign of the derivative

Gradients of tangents can, of course, be negative or zero. They are not always positive.

At a point $(a, g(a))$ on the graph of $y = g(x)$, the gradient of the tangent is $g'(a)$.



Some features of the graph shown are:

- For $x < b$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.
- For $x = b$, the gradient of the tangent is zero, i.e. $g'(b) = 0$.
- For $b < x < a$, the gradient of any tangent is negative, i.e. $g'(x) < 0$.
- For $x = a$, the gradient of the tangent is zero, i.e. $g'(a) = 0$.
- For $x > a$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.

Note: This function g is strictly decreasing on the open interval (b, a), but it is also strictly decreasing on the closed interval $[b, a]$. Similarly, the function g is strictly increasing on the intervals $[a, \infty)$ and $(-\infty, b]$.

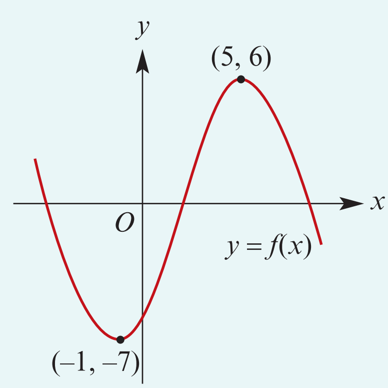
Example 13

For the graph of $y = f(x)$ shown, find the values of x for which:

a $f'(x) > 0$

b $f'(x) < 0$

c $f'(x) = 0$



Solution

a $f'(x) > 0$ for $-1 < x < 5$

b $f'(x) < 0$ for $x < -1$ or $x > 5$

c $f'(x) = 0$ for $x = -1$ or $x = 5$

Section summary

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
- Angle of inclination of tangent
  - A straight line, the tangent, is associated with each point on a smooth curve.
  - If α is the angle that a straight line makes with the positive direction of the x-axis, then the gradient of the line is given by $m = \tan\alpha$.
- Increasing and decreasing functions
  - A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
  - A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
  - If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
  - If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Exercise 7B

Skillsheet

SF

Example 5

1 For each of the following, find the derivative with respect to x:

a x^5 **b** $4x^7$ **c** $6x$
d $5x^2 - 4x + 3$ **e** $4x^3 + 6x^2 + 2x - 4$ **f** $5x^4 + 3x^3$
g $-2x^2 + 4x + 6$ **h** $6x^3 - 2x^2 + 4x - 6$

Example 6

2 For each of the following, find the derivative of $f(x)$ and thus find $f'(1)$:

a $f(x) = 2x^3 - 5x^2 + 1$ **b** $f(x) = -2x^3 - x^2 - 1$
c $f(x) = x^4 - 2x^3 + 1$ **d** $f(x) = x^5 - 3x^3 + 2$

Example 7

3 **a** For the curve determined by the rule $f(x) = 2x^3 - 5x^2 + 2$, find the gradient of the tangent line to the curve at the point $(1, -1)$.

b For the curve determined by the rule $f(x) = -2x^3 - 3x^2 + 2$, find the gradient of the tangent line to the curve at the point $(2, -26)$.

Example 8

4 **a** If $y = t^3$, find $\dfrac{dy}{dt}$.

b If $x = t^3 - t^2$, find $\dfrac{dx}{dt}$.

c If $z = \dfrac{1}{4}x^4 + 3x^3$, find $\dfrac{dz}{dx}$.

Example 9 **5** For each of the following, find $\frac{dy}{dx}$:

a $y = -2x$ **b** $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$ **d** $y = \frac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$ **f** $y = 3x(2x - 4)$

g $y = \frac{10x^7 + 2x^2}{x^2},\ x \neq 0$ **h** $y = \frac{9x^4 + 3x^2}{x},\ x \neq 0$

SF

Example 10 **6** Find:

a $\frac{d}{dx}(2x^2 - 5x^3)$ **b** $\frac{d}{dz}(-2z^2 - 6z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2 + 3)$

d $\frac{d}{dx}(-2x - 5x^3)$ **e** $\frac{d}{dz}(-2z^2 - 6z + 7)$ **f** $\frac{d}{dz}(-z^3 - 4z^2 + 3)$

Example 11 **7** Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = 2x^2 - 4x + 1$, gradient $= -6$ **b** $y = 4x^3$, gradient $= 48$

c $y = x(5 - x)$, gradient $= 1$ **d** $y = x^3 - 3x^2$, gradient $= 0$

Example 12 **8** Find the coordinates of the points on the curve with equation $y = 2x^2 - 3x + 8$ at which the tangent line:

CF

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 2x + 8$.

9 Find the value of x such that the tangent line to the curve $f(x) = x^2 - x$ at $(x, f(x))$:

a makes an angle of 45° with the positive direction of the x-axis

b makes an angle of 135° with the positive direction of the x-axis

c makes an angle of 60° with the positive direction of the x-axis

d makes an angle of 30° with the positive direction of the x-axis

e makes an angle of 120° with the positive direction of the x-axis.

10 For each of the following, find the angle that the tangent line to the curve $y = f(x)$ makes with the positive direction of the x-axis at the given point:

SF

a $y = x^2 + 3x,\ (1, 4)$ **b** $y = -x^2 + 2x,\ (1, 1)$ **c** $y = x^3 + x,\ (0, 0)$

d $y = -x^3 - x,\ (0, 0)$ **e** $y = x^4 - x^2,\ (1, 0)$ **f** $y = x^4 - x^2,\ (-1, 0)$

11 **a** Differentiate $y = (2x - 1)^2$ with respect to x.

CF

b For $y = \frac{x^3 + 2x^2}{x},\ x \neq 0$, find $\frac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} > 0$ for all x.

d Given that $y = \frac{x^3}{3} - x^2 + x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x.

12 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient:

a $y = x^2 + 2x + 1,\ x = 3$
b $y = x^2 - x - 1,\ x = 0$
c $y = 2x^2 - 4x,\ x = -1$
d $y = (2x + 1)(3x - 1)(x + 2),\ x = 4$
e $y = (2x + 5)(3 - 5x)(x + 1),\ x = 1$
f $y = (2x - 5)^2,\ x = 2\frac{1}{2}$

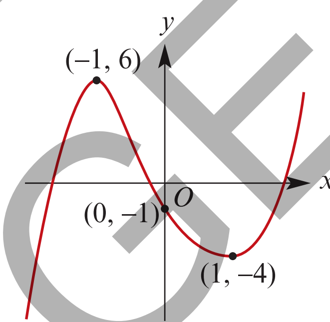
13 For the function $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f'(x) = 10$
f $f(x) = 27$

Example 13

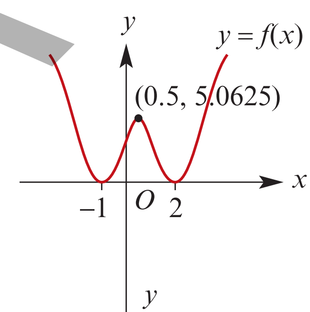
14 For the graph of $y = h(x)$ shown, find the values of x such that:

a $h'(x) > 0$
b $h'(x) < 0$
c $h'(x) = 0$



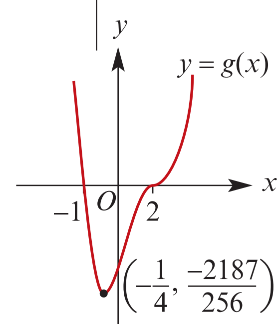
15 For the graph of $y = f(x)$ shown, find the values of x such that:

a $f'(x) > 0$
b $f'(x) < 0$
c $f'(x) = 0$



16 For the graph of $y = g(x)$ shown, find the values of x such that:

a $g'(x) > 0$
b $g'(x) < 0$
c $g'(x) = 0$



17 Find the coordinates of the points on the parabola $y = x^2 - 4x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$.

18 **a** Show that $f(x) = x^3$, $x \in \mathbb{R}$, is a strictly increasing function by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$ and, if $0 > b$, then $f(0) > f(b)$.

b Show that $f(x) = -x^3$, $x \in \mathbb{R}$, is a strictly decreasing function.

SF

CF

CF

19 **a** Show that $f(x) = x^2$, $x \in [0, \infty)$, is a strictly increasing function.
b Show that $f(x) = x^2$, $x \in (-\infty, 0]$, is a strictly decreasing function.

20 For the function $f(x) = x^2 - x - 12$, show that the largest interval for which f is strictly increasing is $[\frac{1}{2}, \infty)$.

21 For each of the following, find the largest interval for which the function is strictly decreasing:

a $y = x^2 + 2x$ **b** $y = -x^2 + 4x$ **c** $y = 2x^2 + 3$ **d** $y = -2x^2 + 6x$

7C Differentiating x^n where n is a negative integer

In the previous sections we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f(x) = x^{-1}$ for $x \neq 0$

$f(x) = 2x + 3 + x^{-2}$ for $x \neq 0$

Example 14

Define the function $f(x) = x^{-3}$ for $x \neq 0$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

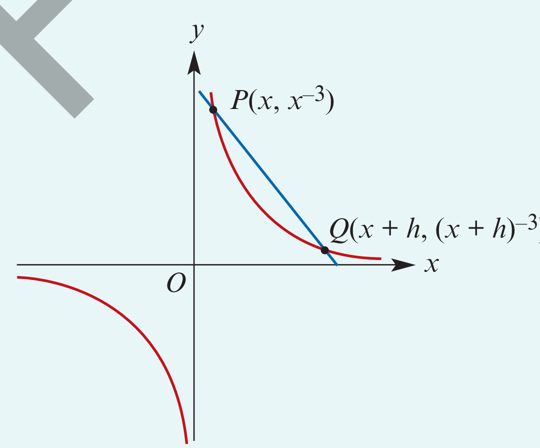
$$\frac{(x+h)^{-3} - x^{-3}}{h}$$

$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

We are now in a position to state a generalisation of the result we found in Section 7B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

If $\quad f(x) = x^4 - 2x^{-3} + x^{-1} + 2 \qquad$ (for $x \neq 0$)

then $\quad f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0$

$\qquad\qquad = 4x^3 + 6x^{-4} - x^{-2} \qquad$ (for $x \neq 0$)

Example 16

Find the derivative of $f(x) = 3x^2 - 6x^{-2} + 1$, $x \neq 0$.

Solution

$f'(x) = 3(2x) - 6(-2x^{-3}) + 0$

$\qquad = 6x + 12x^{-3} \qquad$ (for $x \neq 0$)

Example 17

Find the gradient of the tangent to the curve $y = x^2 + \dfrac{1}{x}$ at the point $(1, 2)$.

Solution

$$\frac{dy}{dx} = 2x + (-x^{-2})$$

$$= 2x - x^{-2}$$

When $x = 1$, $\dfrac{dy}{dx} = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 18

Show that the derivative of the function $f(x) = x^{-3}$, $x \neq 0$, is always negative.

Solution

$f'(x) = -3x^{-4} = -\dfrac{3}{x^4} \quad$ (for $x \neq 0$)

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

Section summary

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Exercise 7C

SF

1 **a** Sketch the graph of $f(x) = \frac{2}{x^2}$, $x \neq 0$.

b Let P be the point $(1, 2)$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

c Hence find the gradient of the tangent to the curve $y = \frac{2}{x^2}$ at $(1, 2)$.

Example 14

2 **a** Let $f(x) = \frac{1}{x - 3}$, $x \neq 3$. Find $f'(x)$ by first principles.

b Let $f(x) = \frac{1}{x + 2}$, $x \neq -2$. Find $f'(x)$ by first principles.

CF

3 Let $f(x) = x^{-4}$. Find $f'(x)$ by first principles.
Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 15, 16

SF

4 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\frac{5}{x^3} + 6x^2$ **c** $\frac{-5}{x^3} + \frac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ **e** $\frac{4x^2 + 2x}{x^2}$

5 Find the derivative of each of the following:

a $\frac{2z^2 - 4z}{z^2}$, $z \neq 0$ **b** $\frac{6 + z}{z^3}$, $z \neq 0$ **c** $16 - z^{-3}$, $z \neq 0$

d $\frac{4z + z^3 - z^4}{z^2}$, $z \neq 0$ **e** $\frac{6z^2 - 2z}{z^4}$, $z \neq 0$ **f** $\frac{6}{x} - 3x^2$, $x \neq 0$

Example 17

6 Find the gradient of the tangent to each of the following curves at the stated point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = x^{-2} - \frac{1}{x}$, $x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \frac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

Example 18

7 Show that the derivative of the function $f(x) = -2x^{-5}$, $x \neq 0$, is always positive.

8 Find the x-coordinates of the points on the curve $y = \frac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

CF

9 Given that the curve $y = ax^2 + \frac{b}{x}$ has a gradient of -5 at the point $(2, -2)$, find the values of a and b.

10 Find the gradient of the curve $y = \dfrac{2x-4}{x^2}$ at the point where the curve crosses the x-axis. SF

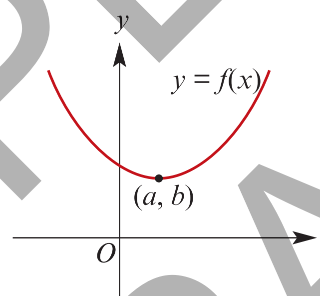
11 The gradient of the curve $y = \dfrac{a}{x} + bx^2$ at the point $(3, 6)$ is 7. Find the values of a and b. CF

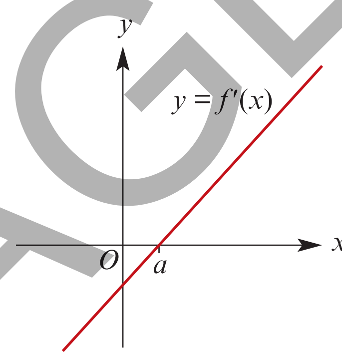
12 For the curve with equation $y = \dfrac{5}{3}x + kx^2 - \dfrac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\frac{1}{2}$ are perpendicular. CU

7D The graph of the derivative function

First consider the quadratic function with rule $y = f(x)$ shown in the graph on the left. The vertex is at the point with coordinates (a, b).

- For $x < a$, $f'(x) < 0$.
- For $x = a$, $f'(x) = 0$.
- For $x > a$, $f'(x) > 0$.



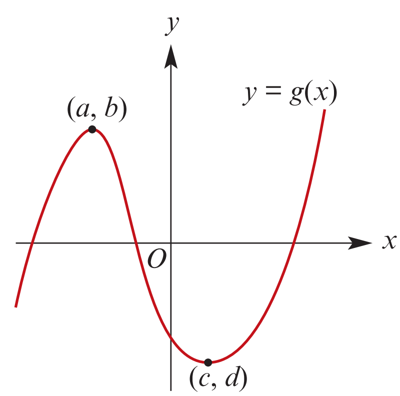


The graph of the derivative function with rule $y = f'(x)$ is therefore as shown on the right.

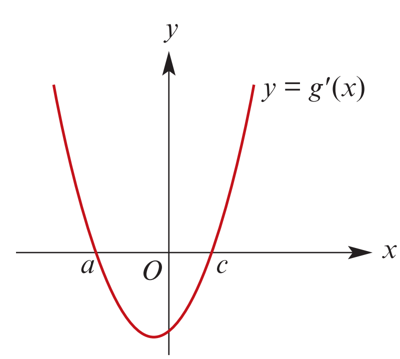
The derivative f' is known to be linear as f is quadratic.

Now consider the cubic function with rule $y = g(x)$ shown in the graph.

- For $x < a$, $g'(x) > 0$.
- For $x = a$, $g'(x) = 0$.
- For $a < x < c$, $g'(x) < 0$.
- For $x = c$, $g'(x) = 0$.
- For $x > c$, $g'(x) > 0$.



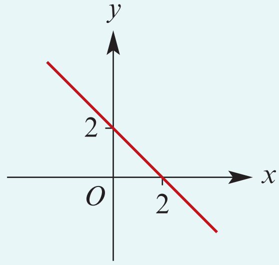
The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative g' is known to be quadratic as g is cubic.



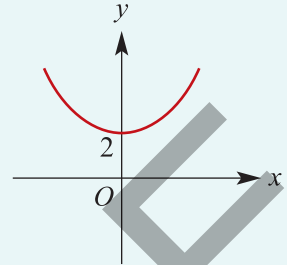
Example 19

Sketch the graph of the derivative function for each of the functions of the graphs shown:

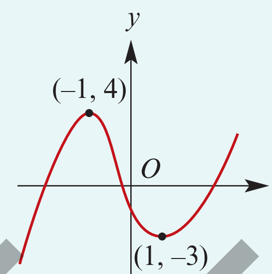
a



b



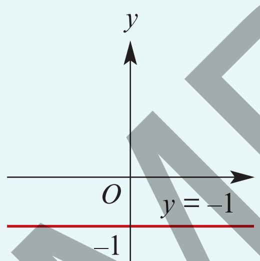
c



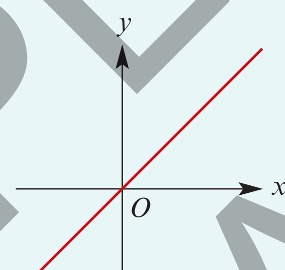
Solution

Note: Not all features of the graphs are known.

a



b



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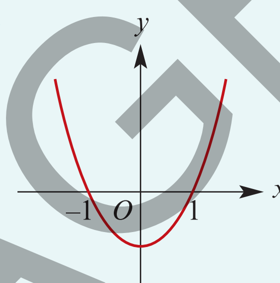


Continuous functions

In this course, we only require an intuitive understanding of continuity. Informally, we say that a function f is continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break.

We can give a more formal definition of continuity using limits as follows.

A function f is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Most of the functions that we study in this course are continuous at each point of their domains. However, we have considered piecewise-defined functions that are not continuous.

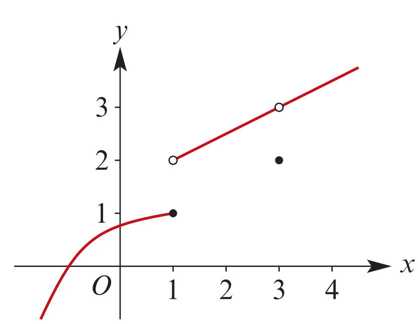
For example, the graph of a piecewise-defined function f is shown on the right. We see that:

- $f(x) \to 1$ as $x \to 1$ from the left
- $f(x) \to 2$ as $x \to 1$ from the right.

Since $f(x)$ does not approach a unique value as $x \to 1$, we say that $\lim_{x \to 1} f(x)$ does not exist.

The function f has a discontinuity at $x = 1$.

There is also a discontinuity at $x = 3$. In this case, we have $\lim_{x \to 3} f(x) = 3$, but $f(3) = 2$.



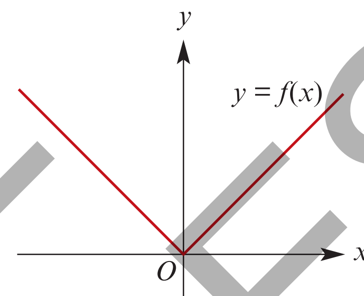
When is a function differentiable?

A function f is said to be **differentiable** at $x = a$ if $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Many of the functions that we study in this course are differentiable at each point of their domains. However, this is not true for all functions.

For example, consider the function

$$f(x) = \sqrt{x^2} = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



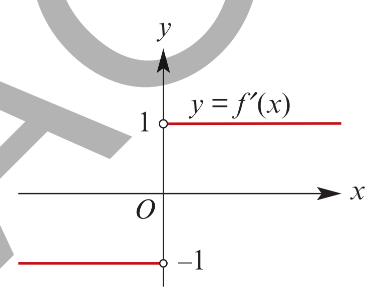
The gradient of the secant through the points (0, 0) and $(0 + h, f(0 + h))$ is given by

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} 1 & \text{for } h > 0 \\ -1 & \text{for } h < 0 \end{cases}$$

The gradient does not approach a unique value as $h \to 0$, and so we say $\lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

The graph of f has gradient 1 to the right of 0, and gradient -1 to the left of 0. Therefore the derivative function is given by

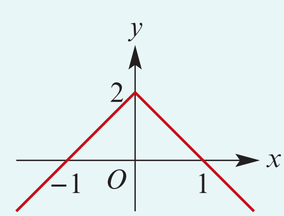
$$f'(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$



The domain of f' is $\mathbb{R} \setminus \{0\}$. The graph of f' is shown on the right.

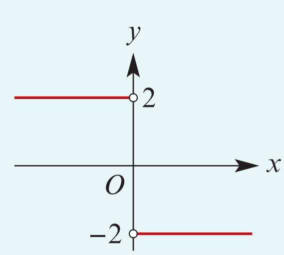
Example 20

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$,
i.e. the function is not differentiable at $x = 0$.

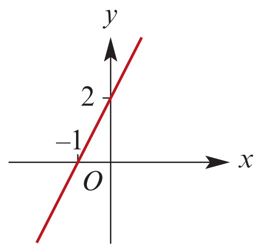


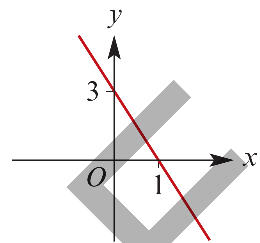
Note: If a function is differentiable at $x = a$, then it is also continuous at $x = a$. However, the converse is not true. For example, the function f from Example 20 is continuous at $x = 0$, but not differentiable at $x = 0$.

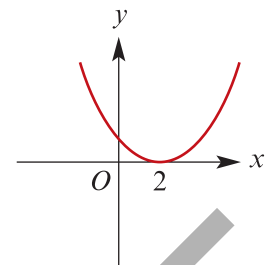
Exercise 7D

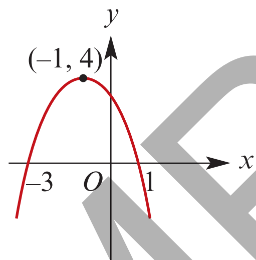
SF

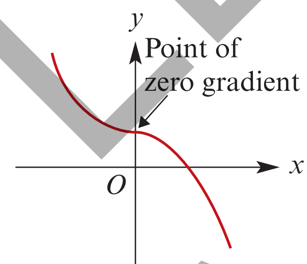
Example 19 **1** Sketch the graph of the derivative function for each of the following functions:

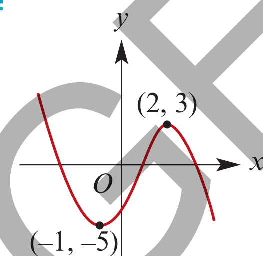
a


b


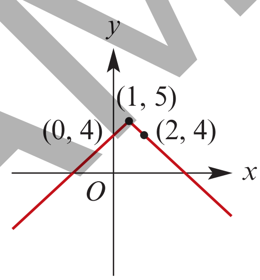
c


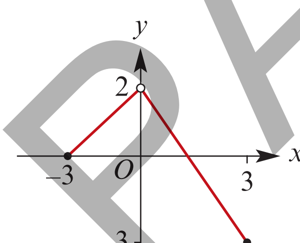
d


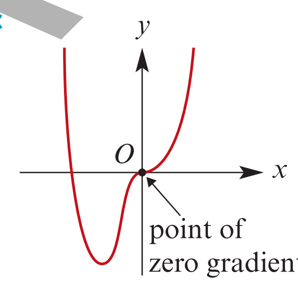
e


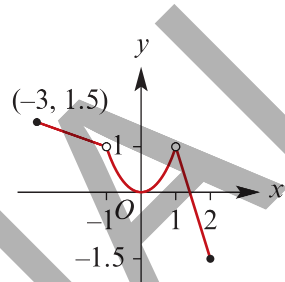
f


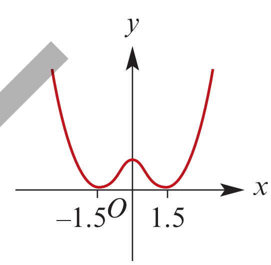
Example 20 **2** Sketch the graph of the derivative function for each of the following functions:

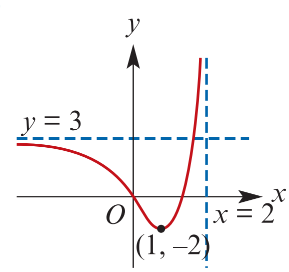
a


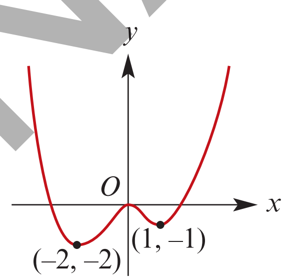
b


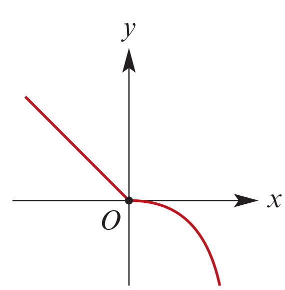
c


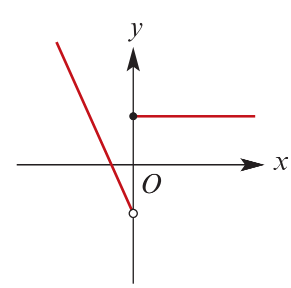
d


e


f


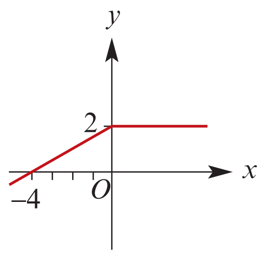
g


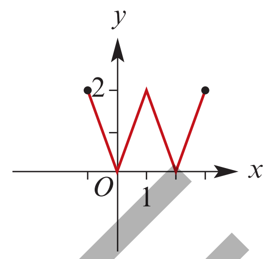
h


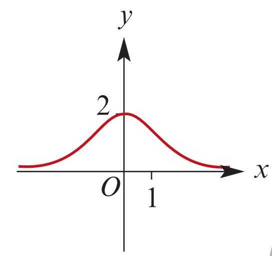
i


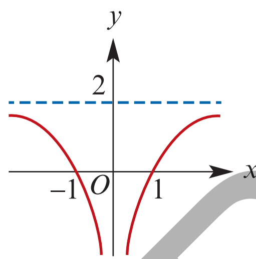
SF

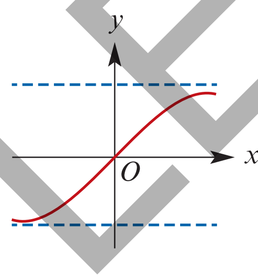
3 Match the graphs of the functions **a**–**f** with the graphs of their derivatives **A**–**F**:

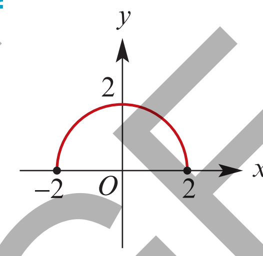
a


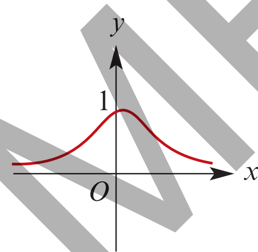
b


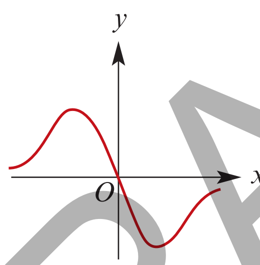
c


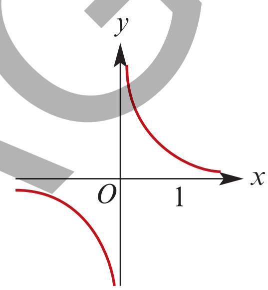
d


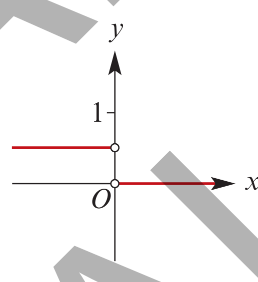
e


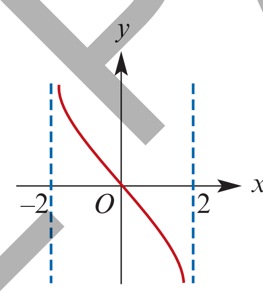
f


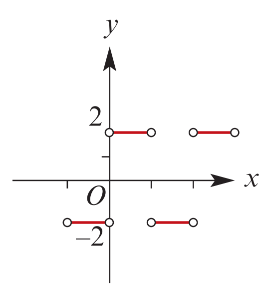
A


B


C


D


E


F


4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$.

b Using the same screen, plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \leq x \leq 1$, find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum.

5 For each of the following functions, plot the graphs of the function and its derivative on the same screen. Comment.

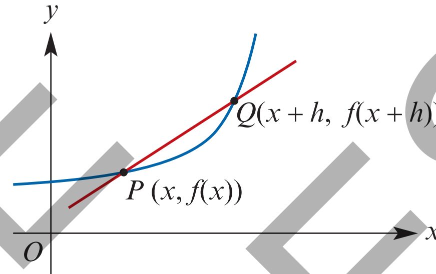
a $f(x) = \dfrac{x^3}{3} - x^2 + x + 1$ **b** $g(x) = x^3 + 2x + 1$

Review

Chapter summary

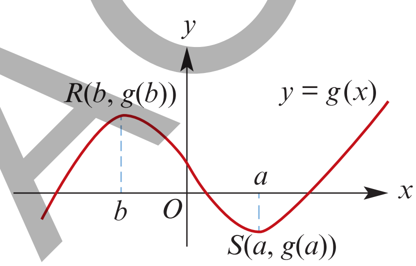
The derivative

- The notation for the limit as h approaches 0 is $\lim_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
  $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
  $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Basic derivatives

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.
- For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Rules for differentiation

- If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.

Technology-free questions

1 Points P and Q lie on the curve $y = x^3$. The x-coordinates of P and Q are 2 and $2 + h$ respectively.

a Find the gradient of the secant PQ.

b Hence find the gradient of the tangent to the curve $y = x^3$ at the point P.

SF

2 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval $[3, 5]$.

b Find the instantaneous rate of change of y with respect to x when $x = -4$.

3 Find the derivative of each of the following functions:

a $f(x) = 5 - 3x$ **b** $f(x) = 4x^2 - 2x + 11$ **c** $f(x) = \frac{1}{2}x(x + 3)$

d $f(x) = \frac{5}{x^2} - \frac{3}{x} + x^2$ **e** $f(x) = \frac{x^3 + 4}{x}$ **f** $f(x) = \frac{(x^2 - 1)(x^2 - 5)}{x^4}$

4 Let $f(x) = 2x^2 - 3x + 5$. Find:

a $f'(x)$ **b** $f'(0)$ **c** the value of x such that $f'(x) = 1$

5 Let $f(x) = x^3 + 3x^2 - 1$. Find the values of x for which:

a $f'(x) = 0$ **b** $f'(x) > 0$ **c** $f'(x) < 0$

6 Let $y = 1 - x^2$. Prove that $x\frac{dy}{dx} + 2 = 2y$ for all values of x.

7 Let $A = 4\pi r^2$. Calculate the value of $\frac{dA}{dr}$ when $r = 3$.

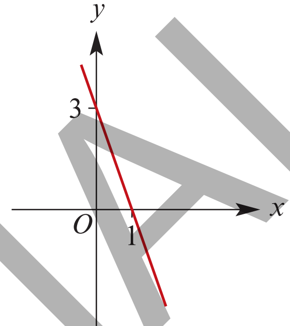
8 At what point on the graph of $y = 1.8x^2$ is the gradient equal to 1?

9 If $y = 3x^2 - 4x + 7$, find the value of x such that $\frac{dy}{dx} = 0$.

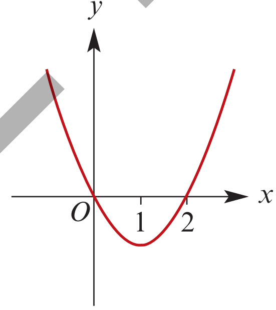
10 Let $y = x^4$. Prove that $x\frac{dy}{dx} = 4y$.

11 Sketch the graph of the derivative function for each of the following functions:

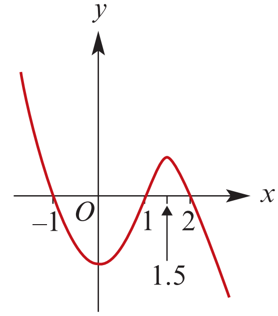
a



b



c



12 Let $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

13 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

14 The graph of $y = bx^2 - cx$ crosses the x-axis at the point $(4, 0)$. The gradient at this point is 1. Find the values of b and c.

Review

Multiple-choice questions

1 If $f(x) = \dfrac{4x^4 - 12x^2}{3x}$, then $f'(x)$ is equal to

A $\dfrac{16x^3 - 24x}{3}$ **B** $4x^2 - 4$ **C** $\dfrac{16x^3 - 24x}{3x}$ **D** $4x^2 - 8x$ **E** $\dfrac{8x^2 - 16x}{3x}$

2 Let $f(x) = x^5 + x^3 + x$. The value of $f'(1)$ is

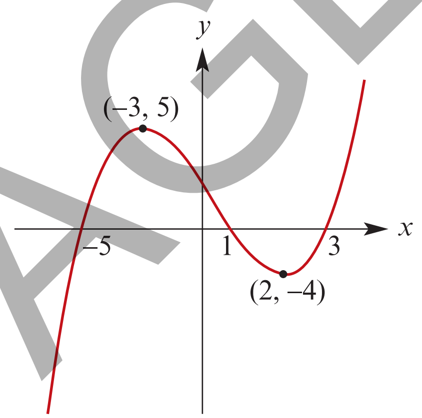
A 0 **B** 1 **C** 2 **D** 9 **E** −3

3 If $f(x) = \dfrac{3}{x}$, then $\dfrac{f(x+h) - f(x)}{h}$ is equal to

A $\dfrac{-3}{x(x+h)}$ **B** $\dfrac{3}{x^2}$ **C** $\dfrac{-3}{x^2}$ **D** $\dfrac{-3}{h(x+h)}$ **E** $f'(x)$

4 For the graph shown, the gradient is positive for

A $-3 < x < 2$
B $-3 \leq x \leq 2$
C $x < -3$ or $x > 2$
D $x \leq -3$ or $x \geq 2$
E $-3 \leq x \leq 3$



5 If $f(x) = 4x(2 - 3x)$, then $f'(x) < 0$ for

A $x < \dfrac{1}{3}$ **B** $0 < x < \dfrac{2}{3}$ **C** $x = \dfrac{1}{3}$ **D** $x > \dfrac{1}{3}$ **E** $x = 0, \dfrac{2}{3}$

6 The point on the curve defined by the equation $y = (x + 3)(x - 2)$ at which the gradient is −7 has coordinates

A (−4, 6) **B** (−4, 0) **C** (−3, 0) **D** (−3, −5) **E** (−2, 0)

7 If $\dfrac{f(3+h) - f(3)}{h} = h^2 + 9h + 27$, then $f'(3)$ equals

A 3 **B** 9 **C** 18 **D** 27 **E** 63

8 The function $y = ax^2 - bx$ has zero gradient only for $x = 2$. The x-axis intercepts of the graph of this function are

A $\frac{1}{2}, -\frac{1}{2}$ **B** 0, 4 **C** 0, −4 **D** $0, \frac{1}{2}$ **E** $0, -\frac{1}{2}$

9 The function $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing only when

A $x > 0$ **B** $-3 < x < 1$ **C** $x < -1$ or $x > 3$
D $x < -3$ or $x > 1$ **E** $-1 < x < 3$