Chapter 2

2 Coordinate geometry and transformations

Objectives

- To revise coordinate geometry:
  - finding the **distance** between two points
  - finding the **midpoint** of a line segment
  - calculating the **gradient** of a straight line
  - interpreting and using different forms of the **equation of a straight line**
  - finding the **angle of slope** of a straight line
  - determining the gradient of a line **perpendicular** to a given line.
- To introduce a notation for considering transformations of the plane, including **translations**, **dilations** from an axis and **reflections** in an axis.
- To use transformations to help with graph sketching.
- To consider transformations of power functions.
- To determine the rule for a function given sufficient information.

Much of the material presented in this chapter has been covered in Mathematical Methods Units 1 & 2. The chapter provides a framework for revision with worked examples and exercises.

Many graphs of functions can be described as transformations of graphs of other functions, or 'movements' of graphs about the Cartesian plane. For example, the graph of the function $y = -x^2$ can be considered as a reflection in the x-axis of the graph of the function $y = x^2$.

A good understanding of transformations, combined with knowledge of the 'simplest' function and its graph in each family, provides an important tool with which to sketch graphs and identify rules of more complicated functions.

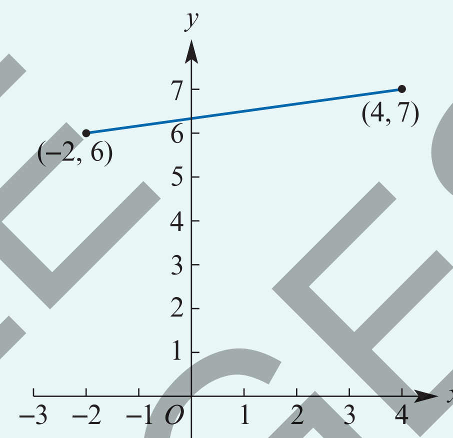
2A Linear coordinate geometry

In this section, we revise the concepts of linear coordinate geometry.

Example 1

A straight line passes through the points $A(-2, 6)$ and $B(4, 7)$. Find:

- **a** the distance AB
- **b** the midpoint of line segment AB
- **c** the gradient of line AB
- **d** the equation of line AB
- **e** the equation of the line parallel to AB which passes through the point $(1, 5)$
- **f** the equation of the line perpendicular to AB which passes through the midpoint of AB.



Solution

a The distance AB is

$$\sqrt{(4-(-2))^2+(7-6)^2}=\sqrt{37}$$

Explanation: The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

b The midpoint of AB is

$$\left(\frac{-2+4}{2}, \frac{6+7}{2}\right)=\left(1, \frac{13}{2}\right)$$

Explanation: The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

c The gradient of line AB is

$$\frac{7-6}{4-(-2)}=\frac{1}{6}$$

Explanation: Gradient

$$m=\frac{y_2-y_1}{x_2-x_1}$$

d The equation of line AB is

$$y-6=\frac{1}{6}(x-(-2))$$

which simplifies to $6y-x-38=0$.

Explanation: Equation of a straight line passing through a given point (x_1, y_1) and having gradient m is $y-y_1=m(x-x_1)$.

e Gradient $m=\frac{1}{6}$ and $(x_1, y_1)=(1, 5)$.

The line has equation

$$y-5=\frac{1}{6}(x-1)$$

which simplifies to $6y-x-29=0$.

Explanation: Parallel lines have the same gradient.

f A perpendicular line has gradient -6.

Thus the equation is

$$y-\frac{13}{2}=-6(x-1)$$

which simplifies to $2y+12x-25=0$.

Explanation: If two straight lines are perpendicular to each other, then the product of their gradients is -1.

Example 2

A fruit and vegetable wholesaler sells 30 kg of hydroponic tomatoes for \$148.50 and sells 55 kg of hydroponic tomatoes for \$247.50. Find a linear model for the cost, \$$C$, of x kg of hydroponic tomatoes. How much would 20 kg of tomatoes cost?

Solution

Let $(x_1, C_1) = (30, 148.5)$ and $(x_2, C_2) = (55, 247.5)$.

The equation of the straight line is given by

$$C - C_1 = m(x - x_1) \quad \text{where } m = \frac{C_2 - C_1}{x_2 - x_1}$$

Now $m = \dfrac{247.5 - 148.5}{55 - 30} = 3.96$ and so

$$C - 148.5 = 3.96(x - 30)$$

Therefore the straight line has equation $C = 3.96x + 29.7$.

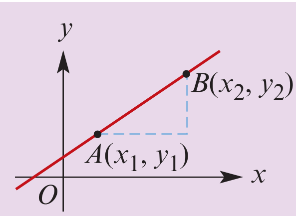
Substitute $x = 20$:

$$C = 3.96 \times 20 + 29.7 = 108.9$$

Hence it would cost \$108.90 to buy 20 kg of tomatoes.

The following is a summary of the material that is assumed to have been covered in Mathematical Methods Units 1 & 2.

Section summary



- For two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
  - The **distance** between points A and B is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
  - The **midpoint** of the line segment AB is the point with coordinates $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
  - The **gradient** of the line AB is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\dfrac{x}{a} + \dfrac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- The **angle of slope** of a straight line is found using $m = \tan\theta$, where m is the gradient and θ is the angle that the line makes with the positive direction of the x-axis.
- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Exercise 2A

Skillsheet
Example 1

1 A straight line passes through the points $A(-2, 6)$ and $B(4, -7)$. Find:

a the distance AB **b** the midpoint of line segment AB
c the gradient of line AB **d** the equation of line AB
e the equation of the line parallel to AB which passes through the point $(1, 5)$
f the equation of the line perpendicular to AB which passes through the midpoint of AB.

2 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(1, 4)$, $B(5, 11)$ **b** $A(-6, 4)$, $B(1, -8)$ **c** $A(-1, -6)$, $B(4, 7)$

3 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-4, 5)$, $M(0, 6)$ **b** $X(-1, -4)$, $M(2, -3)$ **c** $X(6, -3)$, $M(4, 8)$

4 Use $y = mx + c$ to sketch the graph of each of the following:

a $y = 3x - 3$ **b** $y = -3x + 4$ **c** $3y + 2x = 12$
d $4x + 6y = 12$ **e** $3y - 6x = 18$ **f** $8x - 4y = 16$

5 Find the equations of the following straight lines:

a gradient +2, passing through $(4, 2)$ **b** gradient −3, passing through $(-3, 4)$
c passing through $(1, 3)$ and $(4, 7)$ **d** passing through $(-2, -3)$ and $(2, 5)$

6 Use the intercept form $\frac{x}{a} + \frac{y}{b} = 1$ to find the equation of the straight line passing through:

a $(-3, 0)$ and $(0, 2)$ **b** $(4, 0)$ and $(0, 6)$ **c** $(0, -2)$ and $(6, 0)$

7 Write the following in intercept form and hence draw their graphs:

a $3x + 6y = 12$ **b** $4y - 3x = 12$ **c** $\frac{3}{2}x - 3y = 9$

Example 2

8 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$46 for printing 800 sheets. Find a linear model for the charge, \$$C$, for printing n sheets. How much would they charge for printing 1000 sheets?

9 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.
b Was there a sum already on the register when counting began?
c If so, how much?

10 Find the distance between each of the following pairs of points:

a $(2, 6)$, $(3, 4)$ **b** $(-2, -6)$, $(2, -8)$ **c** $(0, 4)$, $(3, 0)$

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11 **a** Find the equation of the straight line which passes through the point $(1, 6)$ and is:

i parallel to the line with equation $y = 2x + 3$

ii perpendicular to the line with equation $y = 2x + 3$.

b Find the equation of the straight line which passes through the point $(2, 3)$ and is:

i parallel to the line with equation $4x + 2y = 10$

ii perpendicular to the line with equation $4x + 2y = 10$.

12 Find the equation of the line which passes through the point of intersection of the lines $y = x$ and $x + y = 6$ and which is perpendicular to the line with equation $3x + 6y = 12$.

13 The length of the line segment joining $A(2, -1)$ and $B(5, y)$ is 5 units. Find y.

14 For each of the following, find the angle that the line joining the given points makes with the positive direction of the x-axis:

a $(-4, 1),\ (4, 6)$ **b** $(2, 3),\ (-4, 6)$ **c** $(5, 1),\ (-1, -8)$ **d** $(-4, 2),\ (2, -8)$

15 Find the acute angle between the lines $y = 2x + 4$ and $y = -3x + 6$.

16 Given the points $A(a, 3)$, $B(-2, 1)$ and $C(3, 2)$, find the possible values of a if the length of AB is twice the length of BC.

17 Three points have coordinates $A(1, 7)$, $B(7, 5)$ and $C(0, -2)$. Find:

a the equation of the perpendicular bisector of AB

b the point of intersection of this perpendicular bisector and BC.

18 The point (h, k) lies on the line $y = x + 1$ and is 5 units from the point $(0, 2)$. Write down two equations connecting h and k and hence find the possible values of h and k.

19 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x- and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and the point R has x-coordinate $2a$, where $a > 0$.

a Find the y-coordinate of R in terms of a.

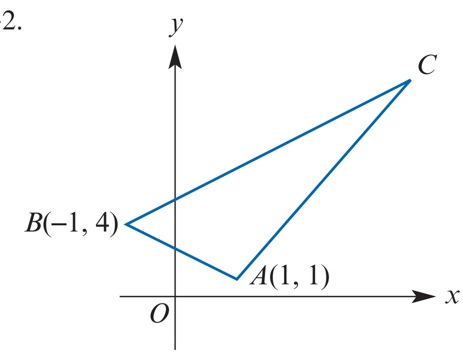
b Find the value of a if the gradient of PR is -2.

20 The figure shows a triangle ABC with $A(1, 1)$ and $B(-1, 4)$. The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.

a Find the value of m.

b Find the coordinates of C.

c Show that $AC = 2AB$.



21 $ABCD$ is a parallelogram, with vertices labelled anticlockwise, such that A and C are the points $(-1, 5)$ and $(5, 1)$ respectively.

a Find the coordinates of the midpoint of AC.

b Given that BD is parallel to the line $y + 5x = 2$, find the equation of BD.

c Given that BC is perpendicular to AC, find:

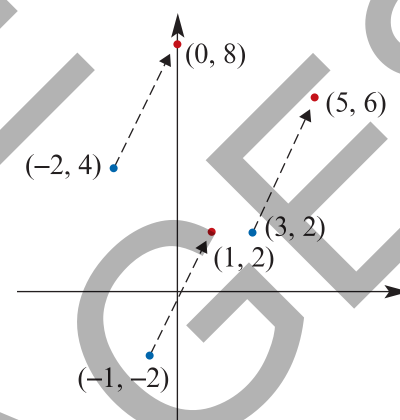
i the equation of BC **ii** the coordinates of B **iii** the coordinates of D.

2B Translations

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

▶ Notation

Consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up).



This can be described by the rule $(x, y) \to (x + 2, y + 4)$. This reads as '$(x, y)$ maps to $(x + 2, y + 4)$'.

For example, $(3, 2) \to (3 + 2, 2 + 4)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). We can write:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For positive numbers h and k:

- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

  $$(x, y) \to (x + h, y + k)$$

  or $\quad x' = x + h \quad \text{and} \quad y' = y + k$

- A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

  $$(x, y) \to (x - h, y - k)$$

  or $\quad x' = x - h \quad \text{and} \quad y' = y - k$

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(a - h, b - k) \to (a, b)$ under this translation.)

▶ Applying translations to sketch graphs

A translation moves every point on the graph the same distance in the same direction.

Every translation of the plane can be described by giving two components:

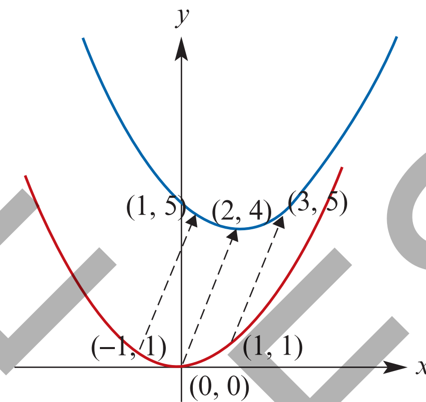
- a translation parallel to the x-axis and
- a translation parallel to the y-axis.

For example, consider a translation of 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis applied to the graph of $y = x^2$.

The set of points $\{ (x, y) : y = x^2 \}$ is translated according to the rule

$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$



For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.

This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.

Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$\Leftrightarrow \; f(a) = b$

$\Leftrightarrow \; f(a + h - h) = b$

$\Leftrightarrow \; f(a + h - h) = b + k - k$

$\Leftrightarrow \; (a + h, b + k)$ is a point on the graph of $y - k = f(x - h)$

Note: The double arrows indicate that the steps are reversible.

Example 3

Find the image of the curve with equation $y = \dfrac{1}{x}$ under a translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

Let (x', y') be the image of the point (x, y).

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = \dfrac{1}{x}$ is mapped to $y' + 2 = \dfrac{1}{x' - 3}$.

The equation of the image can be written as

$$y = \frac{1}{x - 3} - 2$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into the equation $y = \dfrac{1}{x}$.

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, to sketch the graph of $y = \sqrt{x-2}$, note that it is of the form $y = f(x-2)$, where $f(x) = \sqrt{x}$. The graph of $y = \sqrt{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x-2) = (x-2)^2$$

$$f(x) = \frac{1}{x} \qquad f(x-2) = \frac{1}{x-2}$$

Section summary

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x+h, y+k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Exercise 2B

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1 Find the image of the point $(-2, 5)$ after a mapping of a translation:

a of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

b of 3 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis

c defined by the rule $(x, y) \to (x-3, y+2)$

d defined by the rule $(x, y) \to (x-1, y+1)$.

Example 3

2 Find the image of the curve with equation $y = \frac{1}{x}$ under:

a a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b a translation of 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

c a translation of $\frac{1}{2}$ unit in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

3 Sketch the graph of each of the following. Label asymptotes and axis intercepts, and state the domain and range.

a $y = \frac{1}{x} + 3$ **b** $y = \frac{1}{x^2} - 3$ **c** $y = \frac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \frac{1}{x-1}$ **f** $y = \frac{1}{x} - 4$

g $y = \frac{1}{x+2}$ **h** $f(x) = \frac{1}{(x-3)^2}$ **i** $f(x) = \frac{1}{x-1} + 1$

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4 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$
d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

5 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = x^2, \quad f_1(x) = (x + 5)^2$ **b** $f(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{x} + 2$

c $f(x) = \frac{1}{x^2}, \quad f_1(x) = \frac{1}{x^2} + 4$ **d** $f(x) = \frac{1}{x^2} - 3, \quad f_1(x) = \frac{1}{x^2}$

e $f(x) = \frac{1}{x - 3}, \quad f_1(x) = \frac{1}{x}$

6 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a translation of 7 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis
ii a translation of 2 units in the negative direction of the x-axis and 6 units in the negative direction of the y-axis
iii a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
iv a translation of 1 unit in the negative direction of the x-axis and 4 units in the positive direction of the y-axis.

a $y = x^{\frac{1}{4}}$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$

7 Find the equation for the image of the graph of each of the following under the stated translation:

a $y = (x - 2)^2 + 3$ Translation: $(x, y) \to (x - 3, y + 2)$
b $y = 2(x + 3)^2 + 3$ Translation: $(x, y) \to (x + 3, y - 3)$
c $y = \frac{1}{(x - 2)^2} + 3$ Translation: $(x, y) \to (x + 4, y - 2)$

8 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}, \quad f_1(x) = \frac{1}{(x - 2)^2} + 3$ **b** $f(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{x + 2} - 3$

c $f(x) = \sqrt{x}, \quad f_1(x) = \sqrt{x + 4} + 2$

2C Dilations and reflections

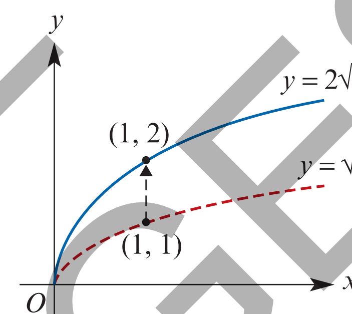
We can determine the equation of the image of a curve under a dilation or a reflection by following the same approach used for translations.

Dilation from the x-axis

A dilation of factor 2 from the x-axis is defined by the rule $(x, y) \to (x, 2y)$. Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \frac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\frac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.

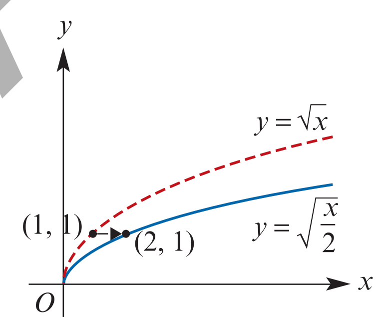


Dilation from the y-axis

A dilation of factor 2 from the y-axis is defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = 2x$ and $y' = y$, and thus $x = \frac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



Example 4

Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 4:

a from the x-axis **b** from the y-axis.

Solution

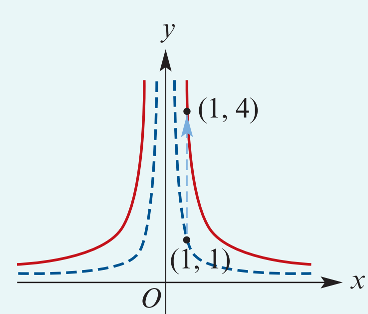
a $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x, \ y' = 4y$.

Rearranging gives $x = x', \ y = \frac{y'}{4}$.

Therefore $y = \frac{1}{x^2}$ becomes $\frac{y'}{4} = \frac{1}{(x')^2}$.

The rule of the transformed function is $y = \frac{4}{x^2}$.



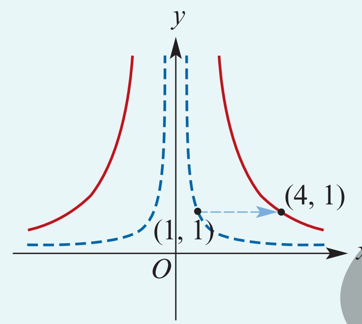
b $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y),
so $x' = 4x$, $y' = y$.

Rearranging gives $x = \dfrac{x'}{4}$, $y = y'$.

Therefore $y = \dfrac{1}{x^2}$ becomes $y' = \dfrac{1}{(\frac{x'}{4})^2}$.

The rule of the transformed function is $y = \dfrac{16}{x^2}$.



Example 5

Determine the factor of dilation when the graph of $y = \sqrt{3x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

Solution

a Note that a dilation from the y-axis 'changes' the x-values. So write the transformed function as

$$y' = \sqrt{3x'}$$

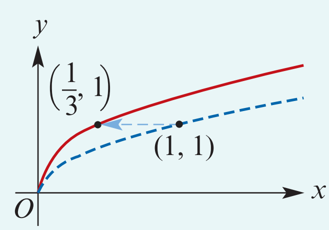
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = 3x'$ and $y = y'$ ('changed' x).

Rearranging gives $x' = \dfrac{x}{3}$ and $y' = y$.

So the mapping is given by $(x, y) \to \left(\dfrac{x}{3}, y\right)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\frac{1}{3}$ from the y-axis to produce the graph of $y = \sqrt{3x}$.



b Note that a dilation from the x-axis 'changes' the y-values. So write the transformed function as

$$\frac{y'}{\sqrt{3}} = \sqrt{x'}$$

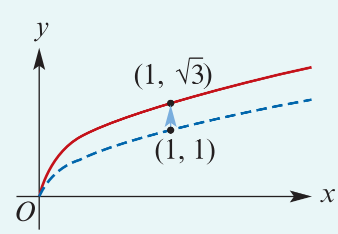
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = x'$ and $y = \dfrac{y'}{\sqrt{3}}$ ('changed' y).

Rearranging gives $x' = x$ and $y' = \sqrt{3}\,y$.

So the mapping is given by $(x, y) \to (x, \sqrt{3}\,y)$.

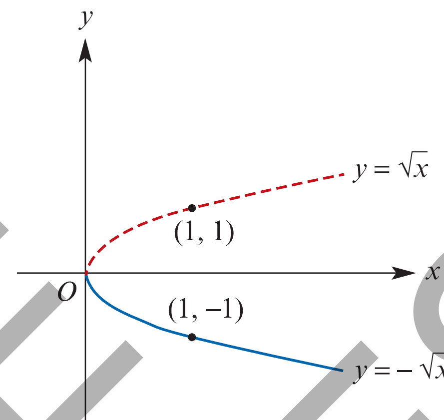
The graph of $y = \sqrt{x}$ is dilated by a factor of $\sqrt{3}$ from the x-axis to produce the graph of $y = \sqrt{3x}$.



▶ Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

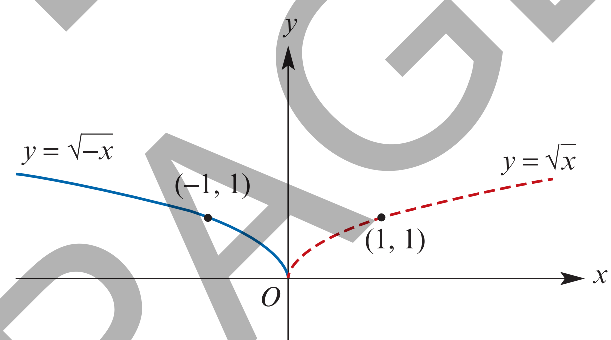
- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



▶ Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



Section summary

Transformations of the graphs of functions

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax,\ y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x,\ y' = by$	$y = bf(x)$

Exercise 2C

Example 4

1 Determine the rule of the image when the graph of $y = \frac{1}{x}$ is dilated by a factor of 3: SF

a from the x-axis **b** from the y-axis.

2 Determine the rule of the image when the graph of $y = \sqrt{x}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

SF

3 Determine the rule of the image when the graph of $y = x^3$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

4 Sketch the graph of each of the following:

a $y = \frac{4}{x}$ **b** $y = \frac{1}{2x}$ **c** $y = \sqrt{3x}$ **d** $y = \frac{2}{x^2}$

5 For $y = f(x) = \frac{1}{x^2}$, sketch the graph of each of the following:

a $y = f(2x)$ **b** $y = 2f(x)$ **c** $y = f\left(\frac{x}{2}\right)$ **d** $y = 3f(x)$

6 Sketch the graphs of each of the following on the one set of axes:

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{3}{2x}$

7 Sketch the graph of the function $f(x) = 3\sqrt{x}$ for $x \in \mathbb{R}^+$.

Example 5

8 Determine the factor of dilation when the graph of $y = \sqrt{5x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

9 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{5}{x^2}$ **b** $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{5x}$

c $f(x) = \sqrt{\frac{x}{3}}$, $f_1(x) = \sqrt{x}$ **d** $f(x) = \frac{1}{4x^2}$, $f_1(x) = \frac{1}{x^2}$

10 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a dilation of factor 4 from the x-axis
ii a dilation of factor $\frac{2}{3}$ from the x-axis
iii a dilation of factor $\frac{1}{2}$ from the y-axis
iv a dilation of factor 5 from the y-axis.

a $y = x^2$ **b** $y = \frac{1}{x^2}$ **c** $y = \sqrt[3]{x}$ **d** $y = \frac{1}{x^4}$ **e** $y = x^{\frac{1}{5}}$

11 Find the equation of the image when the graph of $y = (x - 1)^2$ is reflected:

a in the x-axis **b** in the y-axis.

12 State a transformation which maps the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{-x}$.

13 Find the equation of the image when the graph of each of the functions below is transformed by:

i a reflection in the x-axis
ii a reflection in the y-axis.

a $y = x^3$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$ **e** $y = x^{\frac{1}{4}}$

2D Combinations of transformations

In the previous two sections, we considered three types of transformations separately. In the remainder of this chapter we look at situations where a graph may have been transformed by any combination of dilations, reflections and translations.

For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 6

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y'}{-2} = \sqrt{x'}$$

and hence $y' = -2\sqrt{x'}$. The equation can be written as $y = -2\sqrt{x}$.

b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

So the image of $y = \sqrt{x}$ has equation

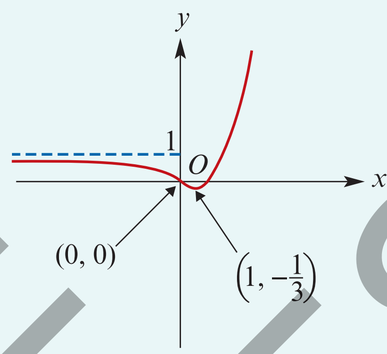
$$\frac{y' + 3}{2} = \sqrt{x' - 2}$$

and hence $y' = 2\sqrt{x' - 2} - 3$. The equation can be written as $y = 2\sqrt{x - 2} - 3$.

Example 7

Sketch the image of the graph shown under the following sequence of transformations:

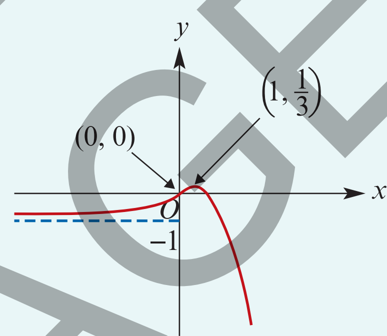
- a reflection in the x-axis
- a dilation of factor 3 from the x-axis
- a translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



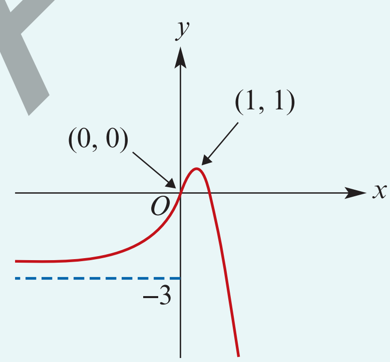
Solution

Consider each transformation in turn and sketch the graph at each stage.

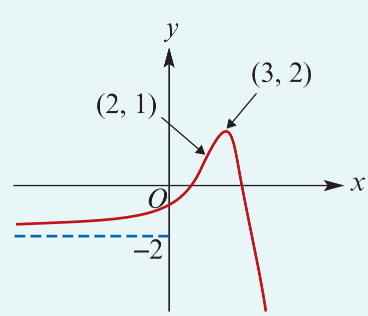
A reflection in the x-axis produces the graph shown on the right.



Next apply the dilation of factor 3 from the x-axis.



Finally, apply the translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



Example 8

For the graph of $y = x^2$:

a Sketch the graph of the image under the sequence of transformations:

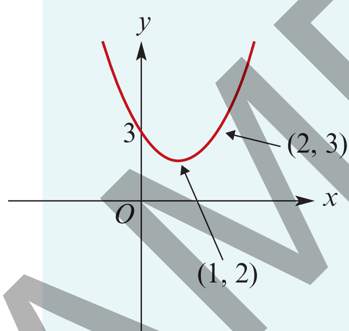
- a translation of 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis
- a dilation of factor 2 from the y-axis
- a reflection in the x-axis.

b State the rule of the image.

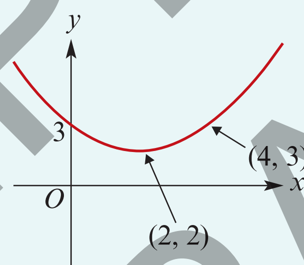
Solution

a Apply each transformation in turn and sketch the graph at each stage.

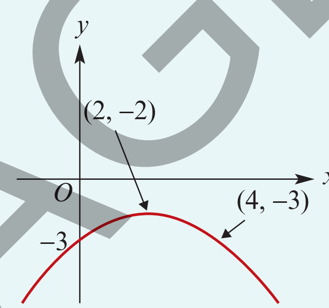
1 The translation:



2 The dilation of factor 2 from the y-axis:



3 The reflection in the x-axis:



b The mapping representing the sequence of transformations is

$$(x, y) \to (x + 1, y + 2) \to (2(x + 1), y + 2) \to (2(x + 1), -(y + 2))$$

Let (x', y') be the image of (x, y). Then $x' = 2(x + 1)$ and $y' = -(y + 2)$.

Rearranging gives $x = \frac{1}{2}(x' - 2)$ and $y = -y' - 2$.

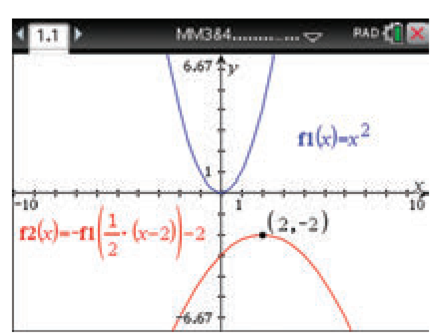
Therefore $y = x^2$ becomes $-y' - 2 = \left(\frac{1}{2}(x' - 2)\right)^2$.

The rule of the image is $y = -\frac{1}{4}(x - 2)^2 - 2$.

TI-84

Using the TI-Nspire CX non-CAS

- In a **Graphs** application, enter $f1(x) = x^2$.
- The rule for the transformed function can be entered as $f2(x) = -f1\left(\frac{1}{2}(x - 2)\right) - 2$.

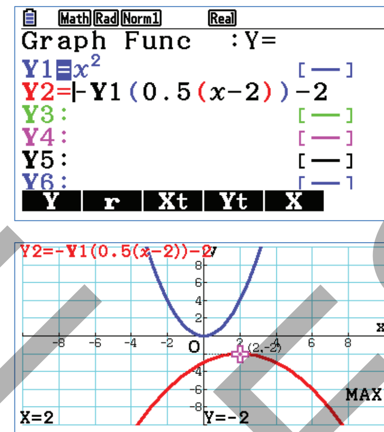


Using the Casio

- Press MENU 5 to select **Graph** mode.
- Enter the rule $y = x^2$ in $Y1$.
- Enter the rule for the transformed function in $Y2$ as shown.

Note: To obtain the function name $Y1$ in the rule for $Y2$, go to the Variable Data menu VARS. Select **Graph** F4, **Y** F1; then press 1.

- Select **Draw** F6 to view the graphs.



Section summary

The following method can be used to find the image of the graph of $y = f(x)$ under a sequence of transformations:

Step 1 Determine the rule $(x, y) \to (x', y')$ for the sequence of transformations.

Step 2 Write down formulas for x' and y' in terms of x and y.

Step 3 Transpose these formulas to express x and y in terms of x' and y'.

Step 4 Substitute these expressions for x and y into the equation $y = f(x)$.

Exercise 2D

Skillsheet
Example 6

1 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

ii a dilation of factor 3 from the y-axis, followed by a translation 2 units in the negative direction of the x-axis and 4 units in the negative direction of the y-axis

iii a dilation of factor 2 from the x-axis, followed by a reflection in the y-axis.

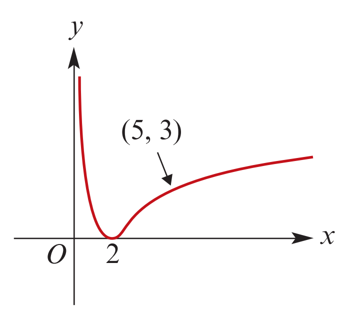
a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x^2}$

CF

Example 7

2 Sketch the image of the graph shown under the following sequence of transformations:

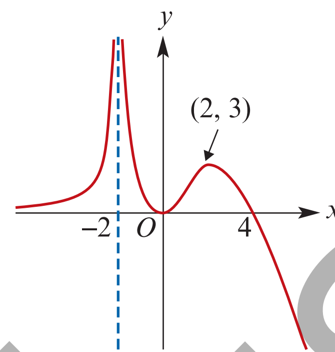
- a reflection in the x-axis
- a dilation of factor 2 from the x-axis
- a translation 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.



CF

3 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the y-axis
- a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis
- a dilation of factor 2 from the y-axis.



4 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

ii a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis

iii a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

iv a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis

v a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis

vi a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x}$ **d** $y = \dfrac{1}{x^3}$ **e** $y = x^{-2}$

5 Find the rule of the image when the graph of $y = \sqrt{x}$ is translated 4 units in the negative direction of the x-axis, reflected in the x-axis and dilated by factor 3 from the y-axis.

Example 8 **6** For the graph of $y = \dfrac{3}{x^2}$:

a Sketch the graph of the image under the sequence of transformations:

- a dilation of factor 2 from the x-axis
- a translation of 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis
- a reflection in the x-axis.

b State the rule of the image.

2E Using transformations to sketch graphs

By considering a rule for a graph as a combination of transformations of a more 'simple' rule, we can readily sketch graphs of many apparently 'complicated' functions.

Example 9

Identify a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{4}{x+5} - 3$. Use this to sketch the graph of $y = \frac{4}{x+5} - 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

Solution

Rearrange the equation of the transformed graph to have the same 'shape' as $y = \frac{1}{x}$:

$$\frac{y' + 3}{4} = \frac{1}{x' + 5}$$

where (x', y') are the coordinates of the image of (x, y).

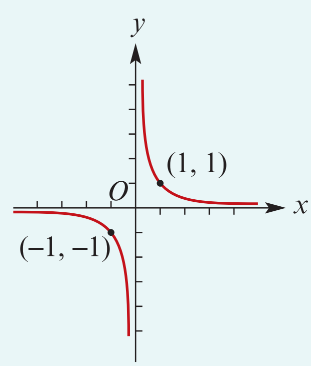
Therefore $x = x' + 5$ and $y = \frac{y' + 3}{4}$. Rearranging gives $x' = x - 5$ and $y' = 4y - 3$.

The mapping is $(x, y) \to (x - 5, 4y - 3)$, and so a sequence of transformations is:

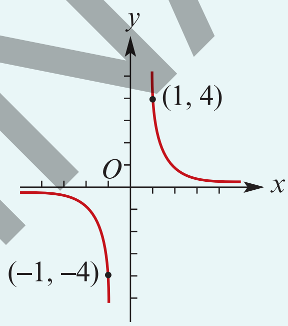
1. a dilation of factor 4 from the x-axis
2. a translation of 5 units in the negative direction of the x-axis
3. a translation of 3 units in the negative direction of the y-axis.

The original graph $y = \frac{1}{x}$ is shown on the right.

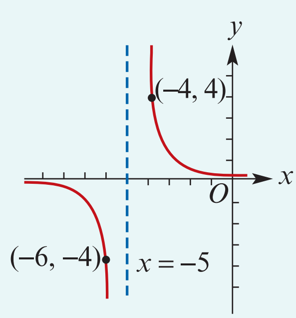
The effect of the transformations is shown below.



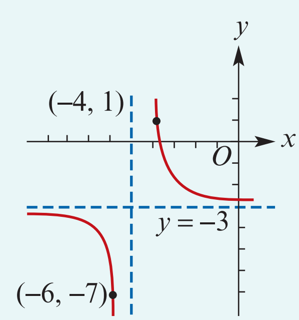
1 Dilation from x-axis:



2 Translation in negative direction of x-axis:



3 Translation in negative direction of y-axis:



Find the axis intercepts in the usual way, as below. The transformed graph, with asymptotes and intercepts marked, is shown on the right.

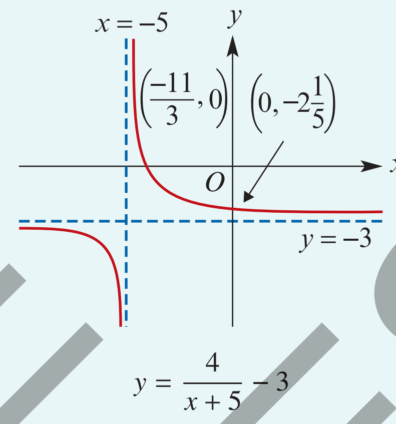
When $x = 0$, $y = \frac{4}{5} - 3 = -2\frac{1}{5}$

When $y = 0$, $\frac{4}{x+5} - 3 = 0$

$$4 = 3x + 15$$

$$3x = -11$$

$$\therefore \quad x = -\frac{11}{3}$$



Once you have done a few of these types of exercises, you can identify the transformations more quickly by carefully observing the rule of the transformed graph and relating it to the 'shape' of the simplest function in its family. Consider the following examples.

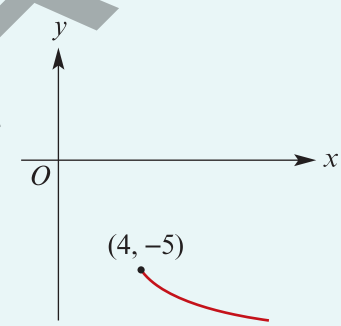
Example 10

Sketch the graph of $y = -\sqrt{x - 4} - 5$.

Solution

The graph is obtained from the graph of $y = \sqrt{x}$ by:

- a reflection in the x-axis, followed by a translation of 5 units in the negative direction of the y-axis, and
- a translation of 4 units in the positive direction of the x-axis.



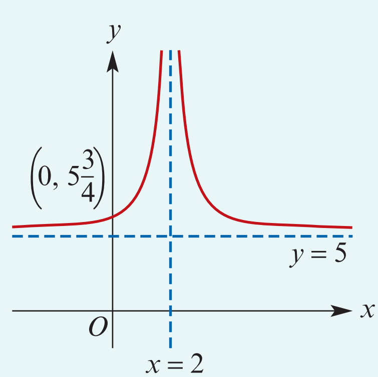
Example 11

Sketch the graph of $y = \frac{3}{(x-2)^2} + 5$.

Solution

This is obtained from the graph of $y = \frac{1}{x^2}$ by:

- a dilation of factor 3 from the x-axis, followed by a translation of 5 units in the positive direction of the y-axis, and
- a translation of 2 units in the positive direction of the x-axis.



Section summary

In general, the function given by the equation

$$y = Af(n(x+c)) + b$$

where $b, c \in \mathbb{R}^+$ and $A, n \in \mathbb{R}^+$, represents a transformation of the graph of $y = f(x)$ by:

- a dilation of factor A from the x-axis, followed by a translation of b units in the positive direction of the y-axis, and
- a dilation of factor $\frac{1}{n}$ from the y-axis, followed by a translation of c units in the negative direction of the x-axis.

Similar statements can be made for $b, c \in \mathbb{R}^-$. The case where $A \in \mathbb{R}^-$ corresponds to a reflection in the x-axis and a dilation from the x-axis. The case where $n \in \mathbb{R}^-$ corresponds to a reflection in the y-axis and a dilation from the y-axis.

Exercise 2E

Skillsheet
Example 9

1 Sketch the graph of each of the following. State the equations of asymptotes and the axis intercepts. State the range of each function.

a $f(x) = \dfrac{3}{x-1}$ **b** $g(x) = \dfrac{2}{x+1} - 1$ **c** $h(x) = \dfrac{3}{(x-2)^2}$

d $f(x) = \dfrac{2}{(x-1)^2} - 1$ **e** $h(x) = \dfrac{-1}{x-3}$ **f** $f(x) = \dfrac{-1}{x+2} + 3$

Example 10, 11

2 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = -\sqrt{x-3}$ **b** $y = -\sqrt{x-3} + 2$ **c** $y = \sqrt{2(x+3)}$

d $y = \dfrac{1}{2x-3}$ **e** $y = 5\sqrt{x+2}$ **f** $y = -5\sqrt{x+2} - 2$

g $y = \dfrac{-3}{x-2}$ **h** $y = \dfrac{-2}{(x+2)^2} - 4$ **i** $y = \dfrac{3}{2x} - 5$

j $y = \dfrac{5}{2x} + 5$ **k** $y = 2(x-3)^2 + 5$ **l** $y = \dfrac{4}{3-x} + 4$

3 Without using your calculator, sketch the graph of $f(x) = \dfrac{3x+2}{x+1}$.

Hint: Show that $f(x) = 3 - \dfrac{1}{x+1}$.

4 Without using your calculator, sketch the graph of $f(x) = \dfrac{4x-5}{2x+1}$.

Hint: Show that $f(x) = 2 - \dfrac{7}{2(x+\frac{1}{2})}$.

SF

2F Transformations of power functions

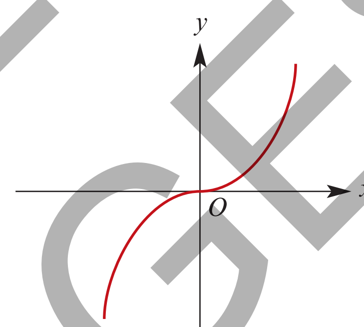
We recall that every quadratic polynomial function can be written in the turning point form $y = a(x - h)^2 + k$. This is not true for polynomials of higher degree. However, there are many polynomials that can be written as $y = a(x - h)^n + k$.

In Section 1E we introduced power functions, which include the functions $f(x) = x^n$, where n is a positive integer. In this section we look at transformations of these functions.

▶ The function $f(x) = x^n$ where n is an odd positive integer

Assume that n is an odd integer with $n \geq 3$. You will recall from Mathematical Methods Units 1 & 2 that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$



Hence the gradient is zero when $x = 0$. Since n is odd and therefore $n - 1$ is even, we have $f'(x) = nx^{n-1} > 0$ for all $x \neq 0$. That is, the gradient of the graph of $y = f(x)$ is positive when $x \neq 0$ and is zero when $x = 0$. Recall that, for functions of this form, the stationary point at (0, 0) is called a stationary point of inflection.

Transformations of $f(x) = x^n$ where n is an odd positive integer

Transformations of these functions result in graphs with rules of the form $y = a(x - h)^n + k$ where a, h and k are real constants.

Example 12

Sketch the graph of:

a $y = (x - 2)^3 + 1$ **b** $y = -(x - 1)^3 + 2$ **c** $y = 2(x + 1)^3 + 2$

Solution

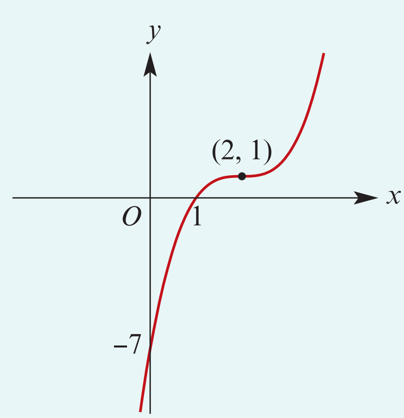
a The translation $(x, y) \to (x + 2, y + 1)$ maps the graph of $y = x^3$ onto the graph of $y = (x - 2)^3 + 1$.

So (2, 1) is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = (-2)^3 + 1 = -7$
- When $y = 0$,

$$0 = (x - 2)^3 + 1$$
$$-1 = (x - 2)^3$$
$$-1 = x - 2$$
$$\therefore \quad x = 1$$



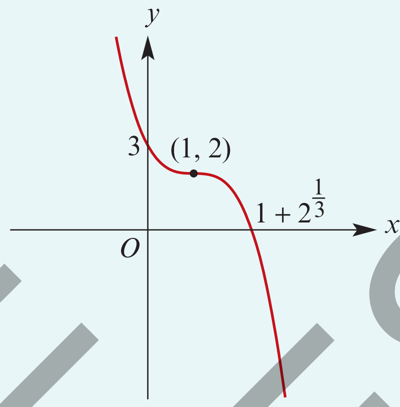
b A reflection in the x-axis followed by the translation $(x, y) \to (x + 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = -(x - 1)^3 + 2$.

So $(1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = -(-1)^3 + 2 = 3$
- When $y = 0$,

$$0 = -(x - 1)^3 + 2$$
$$(x - 1)^3 = 2$$
$$x - 1 = 2^{\frac{1}{3}}$$
$$\therefore \quad x = 1 + 2^{\frac{1}{3}} \approx 2.26$$



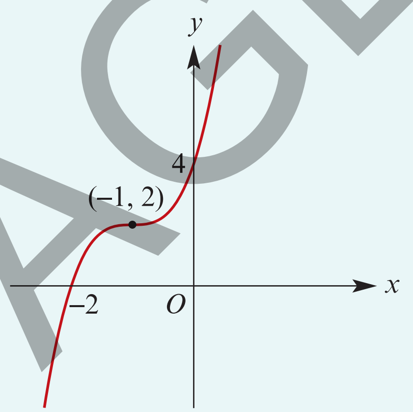
c A dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = 2(x + 1)^3 + 2$.

So $(-1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = 2 + 2 = 4$
- When $y = 0$,

$$0 = 2(x + 1)^3 + 2$$
$$-1 = (x + 1)^3$$
$$-1 = x + 1$$
$$\therefore \quad x = -2$$



Example 13

The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(1, 1)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(1, 1)$ is the point of zero gradient,

$$h = 1 \quad \text{and} \quad k = 1$$

So $y = a(x - 1)^3 + 1$ and, since the graph passes through $(0, 4)$,

$$4 = -a + 1$$
$$\therefore \quad a = -3$$

Example 14

a Find the rule for the image of the graph of $y = x^5$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^5$ to the graph of $y = 6 - 2(x+5)^5$.

Solution

a $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 2, y + 3)$

Let (x', y') be the image of (x, y) under this transformation.

Then $x' = -2x + 2$ and $y' = y + 3$. Hence $x = \dfrac{x' - 2}{-2}$ and $y = y' - 3$.

Therefore the graph of $y = x^5$ maps to the graph of

$$y' - 3 = \left(\frac{x' - 2}{-2}\right)^5$$

i.e. to the graph of

$$y = -\frac{1}{32}(x-2)^5 + 3$$

b Rearrange $y' = 6 - 2(x' + 5)^5$ to $\dfrac{y' - 6}{-2} = (x' + 5)^5$.

Therefore $y = \dfrac{y' - 6}{-2}$ and $x = x' + 5$, which gives $y' = -2y + 6$ and $x' = x - 5$.

The sequence of transformations is:

- reflection in the x-axis
- dilation of factor 2 from the x-axis
- translation 5 units in the negative direction of the x-axis and 6 units in the positive direction of the y-axis.

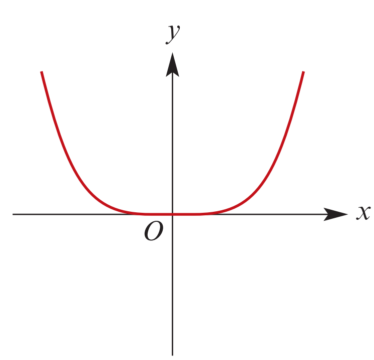
The function $f(x) = x^n$ where n is an even positive integer

Now assume that n is an even integer with $n \geq 2$. The derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is even and therefore $n - 1$ is odd, we have $f'(x) = nx^{n-1} > 0$ for all $x > 0$, and $f'(x) = nx^{n-1} < 0$ for all $x < 0$.

Thus the graph of $y = f(x)$ has a turning point at $(0, 0)$; this point is a local minimum.



Section summary

A graph with rule of the form $y = a(x - h)^n + k$ can be obtained as a transformation of the graph of $y = x^n$.

Exercise 2F

Example 12

1 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

- **a** $f(x) = 2x^3$
- **b** $g(x) = -2x^3$
- **c** $h(x) = x^5 + 1$
- **d** $f(x) = x^3 - 4$
- **e** $f(x) = (x + 1)^3 - 8$
- **f** $f(x) = 2(x - 1)^3 - 2$
- **g** $g(x) = -2(x - 1)^3 + 2$
- **h** $h(x) = 3(x - 2)^3 - 4$
- **i** $f(x) = 2(x - 1)^3 + 2$
- **j** $h(x) = -2(x - 1)^3 - 4$
- **k** $f(x) = (x + 1)^5 - 32$
- **l** $f(x) = 2(x - 1)^5 - 2$

SF

Example 13

2 The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(0, 4)$ and passes through the point $(1, 1)$. Find the values of a, h and k.

CF

3 Find the equation of the image of $y = x^3$ under each of the following transformations:

- **a** a dilation of factor 3 from the x-axis
- **b** a translation with rule $(x, y) \to (x - 1, y + 1)$
- **c** a reflection in the x-axis followed by the translation $(x, y) \to (x + 2, y - 3)$
- **d** a dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y - 2)$
- **e** a dilation of factor 3 from the y-axis.

SF

Example 14

4 **a** Find the rule for the image of the graph of $y = x^3$ under a reflection in the y-axis, followed by a dilation of factor 3 from the y-axis and then a translation 3 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^3$ to the graph of $y = 4 - 3(x + 1)^3$.

CF

5 **a** Find the rule for the image of the graph of $y = x^4$ under a reflection in the y-axis, followed by a dilation of factor 2 from the y-axis and then a translation 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 5 - 3(x + 1)^4$.

6 Sketch the graph of each of the following:

- **a** $y = 3(x - 1)^4 - 2$
- **b** $y = -2(x + 2)^4$
- **c** $y = (x - 2)^4 - 6$
- **d** $y = 2(x - 3)^4 - 1$
- **e** $y = 1 - (x + 4)^4$
- **f** $y = -3(x - 2)^4 - 3$

SF

7 The graph of $y = a(x - h)^4 + k$ has a turning point at $(-2, 3)$ and passes through the point $(0, -6)$. Find the values of a, h and k.

CF

8 The graph of $y = a(x - h)^4 + k$ has a turning point at $(1, 7)$ and passes through the point $(0, 23)$. Find the values of a, h and k.

2G Determining the rule for a function from its graph

Given sufficient information about a curve, we can determine its rule. For example, if we know the coordinates of two points on a hyperbola of the form

$$y = \frac{a}{x} + b$$

then we can find the rule for the hyperbola, i.e. we can find the values of a and b.

Sometimes the rule has a more specific form. For example, the curve may be a dilation of $y = \sqrt{x}$. Then we know its rule is of the form $y = a\sqrt{x}$, and the coordinates of one point on the curve (with the exception of the origin) will be enough to determine the value of a.

Example 15

a The points (1, 5) and (4, 2) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

b The points (2, 1) and (10, 6) lie on a curve with equation $y = a\sqrt{x-1} + b$. Find the values of a and b.

Solution

a When $x = 1$, $y = 5$ and so

$$5 = a + b \qquad (1)$$

When $x = 4$, $y = 2$ and so

$$2 = \frac{a}{4} + b \qquad (2)$$

Subtract (2) from (1):

$$3 = \frac{3a}{4}$$

$$\therefore \quad a = 4$$

Substitute in (1) to find b:

$$5 = 4 + b$$

$$\therefore \quad b = 1$$

The equation of the curve is

$$y = \frac{4}{x} + 1$$

b When $x = 2$, $y = 1$ and so

$$1 = a\sqrt{1} + b$$

i.e. $$1 = a + b \qquad (1)$$

When $x = 10$, $y = 6$ and so

$$6 = a\sqrt{9} + b$$

i.e. $$6 = 3a + b \qquad (2)$$

Subtract (1) from (2):

$$5 = 2a$$

$$\therefore \quad a = \frac{5}{2}$$

Substitute in (1) to find b:

$$1 = \frac{5}{2} + b$$

$$\therefore \quad b = -\frac{3}{2}$$

The equation of the curve is

$$y = \frac{5}{2}\sqrt{x-1} - \frac{3}{2}$$

Exercise 2G

Skillsheet
Example 15a

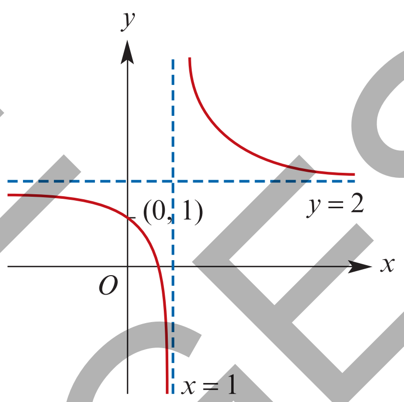
1 The points (1, 4) and (3, 1) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

SF

2 The graph shown has the rule

$$y = \frac{A}{x+b} + B$$

Find the values of A, b and B.



Example 15b

3 The points (3, 1) and (11, 6) lie on a curve with equation $y = a\sqrt{x-2} + b$. Find the values of a and b.

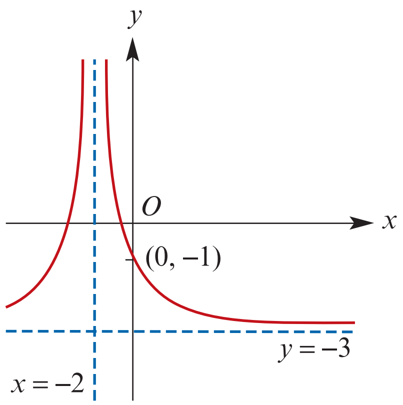
4 The points with coordinates (1, 5) and (16, 11) lie on a curve which has a rule of the form $y = A\sqrt{x} + B$. Find A and B.

5 The points with coordinates (1, 1) and (0.5, 7) lie on a curve which has a rule of the form $y = \frac{A}{x^2} + B$. Find the values of A and B.

6 The graph shown has the rule

$$y = \frac{A}{(x+b)^2} + B$$

Find the values of A, b and B.



7 The points with coordinates (1, −1) and $(2, \frac{3}{4})$ lie on a curve which has a rule of the form $y = \frac{a}{x^3} + b$. Find the values of a and b.

8 The points with coordinates (−1, 4) and (1, −8) lie on a curve which has a rule of the form $y = a\sqrt[3]{x} + b$. Find the values of a and b.

Review

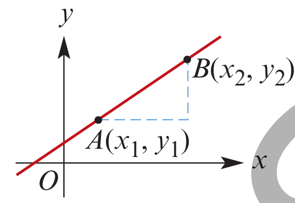
Chapter summary

AS

Nrich

Coordinate geometry

- For two points $A(x_1, y_1)$ and $B(x_2, y_2)$:
  - The **distance** between points A and B is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
  - The **midpoint** of the line segment AB is the point with coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
  - The **gradient** of the line AB is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.



- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\frac{x}{a} + \frac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- The **angle of slope** of a straight line is found using $m = \tan\theta$, where m is the gradient and θ is the angle that the line makes with the positive direction of the x-axis.
- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Transformations of the graphs of functions

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax,\ y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x,\ y' = by$	$y = bf(x)$
Translation	$x' = x + h,\ y' = y + k$	$y - k = f(x - h)$

Technology-free questions

SF

1 Sketch the graphs of the relations:

a $3y + 2x = 5$ **b** $x - y = 6$ **c** $\frac{x}{2} + \frac{y}{3} = 1$

2 Find the distance between the points with coordinates $(-1, 6)$ and $(2, 4)$.

3 Find the midpoint of the line segment AB joining the points $A(4, 6)$ and $B(-2, 8)$.

Review

SF

4 **a** Find the equation of the straight line which passes through (1, 3) and has gradient −2.

b Find the equation of the straight line which passes through (1, 4) and (3, 8).

c Find the equation of the straight line which is perpendicular to the line with equation $y = -2x + 6$ and which passes through the point (1, 1).

d Find the equation of the straight line which is parallel to the line with equation $y = 6 - 2x$ and which passes through the point (1, 1).

5 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-6, 2)$, $M(8, 3)$

b $X(-1, -4)$, $M(2, -8)$

6 The length of the line segment joining $A(5, 12)$ and $B(10, y)$ is 13 units. Find y.

7 Sketch the graph of each of the following. Label any asymptotes and axis intercepts. State the range of each function.

a $f(x) = \frac{1}{x} - 3, \ x \in \mathbb{R} \setminus \{0\}$

b $f(x) = \frac{1}{x-2}, \ x \in (2, \infty)$

c $f(x) = \frac{2}{x-1} - 3, \ x \in \mathbb{R} \setminus \{1\}$

d $f(x) = \frac{-3}{2-x} + 4, \ x \in (2, \infty)$

e $f(x) = 1 - \frac{1}{x-1}, \ x \in \mathbb{R} \setminus \{1\}$

8 Sketch the graph of each of the following:

a $f(x) = 2\sqrt{x-3} + 1$

b $g(x) = \frac{3}{(x-2)^2} - 1$

c $h(x) = \frac{-3}{(x-2)^2} - 1$

9 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = -2(x+1)^3$

b $g(x) = -2(x-1)^5 + 8$

c $h(x) = 2(x-2)^5 + 1$

d $f(x) = 4(x-1)^3 - 4$

10 The points with coordinates (1, 6) and (16, 12) lie on a curve which has a rule of the form $y = a\sqrt{x} + b$. Find a and b.

11 The points with coordinates (1, 3) and (3, 7) lie on a curve with equation of the form $y = \frac{a}{x} + b$. Find the values of a and b.

CF

12 **a** Find the rule for the image of the graph of $y = -x^2$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 4 units in the positive direction of the x-axis and 6 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 6 - 4(x+1)^4$.

Review

CF

13 Identify a sequence of transformations that maps the graph of $y = \frac{1}{x^2}$ onto the graph of $y = \frac{3}{(x-5)^2} + 3$. Use this to sketch the graph of $y = \frac{3}{(x-5)^2} + 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

14 Find a sequence of transformations that takes the graph of $y = 2x^2 - 3$ to the graph of $y = x^2$.

15 Find a sequence of transformations that takes the graph of $y = 2(x-3)^3 + 4$ to the graph of $y = x^3$.

Multiple-choice questions

1 A straight line has gradient $-\frac{1}{2}$ and passes through $(1, 4)$. The equation of the line is

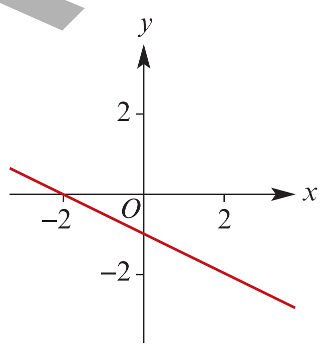
A $y = x + 4$ **B** $y = 2x + 2$ **C** $y = 2x + 4$
D $y = -\frac{1}{2}x + 4$ **E** $y = -\frac{1}{2}x + \frac{9}{2}$

2 The line $y = -2x + 4$ passes through a point $(a, 3)$. The value of a is

A $-\frac{1}{2}$ **B** 2 **C** $-\frac{7}{2}$ **D** -2 **E** $\frac{1}{2}$

3 The gradient of a line that is perpendicular to the line shown could be

A 1 **B** $\frac{1}{2}$ **C** $-\frac{1}{2}$
D 2 **E** -2



4 The coordinates of the midpoint of AB, where A has coordinates $(1, 7)$ and B has coordinates $(-3, 30)$, are

A $(-2, 3)$ **B** $(-1, 8)$ **C** $(-1, 18.5)$ **D** $(-1, 3)$ **E** $(-2, 8.5)$

5 The gradient of the line passing through $(3, -2)$ and $(-1, 10)$ is

A -3 **B** -2 **C** $-\frac{1}{3}$ **D** 4 **E** 3

6 If two lines $-2x + y - 3 = 0$ and $ax - 3y + 4 = 0$ are parallel, then a equals

A 6 **B** 2 **C** $\frac{1}{3}$ **D** $\frac{2}{3}$ **E** -6

7 A straight line passes through $(-1, -2)$ and $(3, 10)$. The equation of the line is

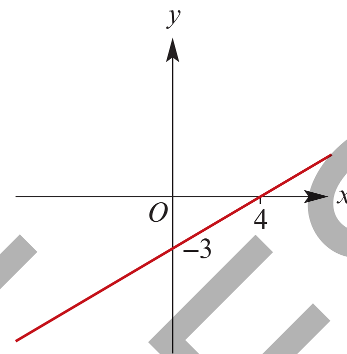
A $y = 3x - 1$ **B** $y = 3x - 4$ **C** $y = 3x + 1$ **D** $y = \frac{1}{3}x + 9$ **E** $y = 4x - 2$

Review

8 The length of the line segment connecting (1, 4) and (5, −2) is

A 10 **B** $2\sqrt{13}$ **C** 12 **D** 50 **E** $2\sqrt{5}$

9 The function with graph as shown has the rule



A $f(x) = 3x - 3$

B $f(x) = -\frac{3}{4}x - 3$

C $f(x) = \frac{3}{4}x - 3$

D $f(x) = \frac{4}{3}x - 3$

E $f(x) = 4x - 4$

10 The midpoint of the line segment joining (0, −6) and (4, d) is

A $\left(-2, \frac{d+6}{2}\right)$ **B** $\left(2, \frac{d+6}{2}\right)$ **C** $\left(\frac{d+6}{2}, 2\right)$ **D** $\left(2, \frac{d-6}{2}\right)$ **E** $\frac{d+6}{4}$

11 The gradient of a line perpendicular to the line through (3, 0) and (0, −6) is

A $\frac{1}{2}$ **B** −2 **C** $-\frac{1}{2}$ **D** 2 **E** 6

12 The point $P(3, -4)$ lies on the graph of a function f. The graph of f is translated 3 units up (parallel to the y-axis) and reflected in the x-axis. The coordinates of the final image of P are

A (6, 4) **B** (3, 1) **C** (3, −1) **D** (−3, 1) **E** (3, 7)

13 The graph of $y = x^3 + 4$ is translated 3 units 'down' and 2 units 'to the right'. The resulting graph has equation

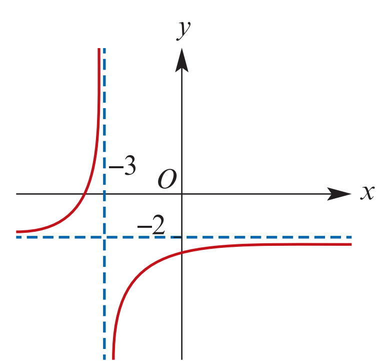
A $y = (x-2)^3 + 2$ **B** $y = (x-2)^3 + 1$ **C** $y = (x-2)^3 + 5$
D $y = (x+2)^3 + 1$ **E** $y = (x+2)^3 + 6$

14 The graph of the function with rule $y = x^2$ is reflected in the x-axis and then translated 4 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis. The rule for the new function is

A $y = (-x+4)^2 - 3$ **B** $y = -(x-4)^2 + 3$ **C** $y = -(x-3)^2 + 4$
D $y = (-x-4)^2 + 3$ **E** $y = -(x+4)^2 - 3$

15 The graph of $y = \frac{a}{x+b} + c$ is shown on the right.

Possible values for a, b and c are



A $a = -1, \ b = 3, \ c = 2$

B $a = 1, \ b = 2, \ c = -3$

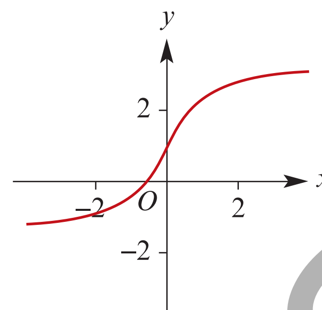
C $a = -1, \ b = -3, \ c = -2$

D $a = -1, \ b = 3, \ c = -2$

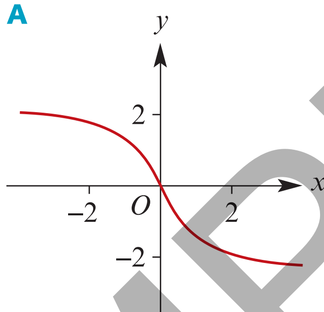
E $a = 1, \ b = 2, \ c = -3$

16 The graph of $y = f(x)$ is shown on the right.

Which one of the following could be the graph of $y = f(-x)$?



A



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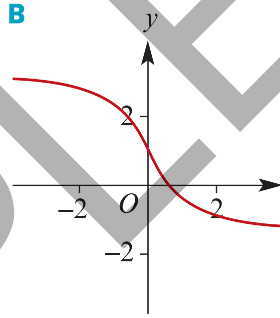

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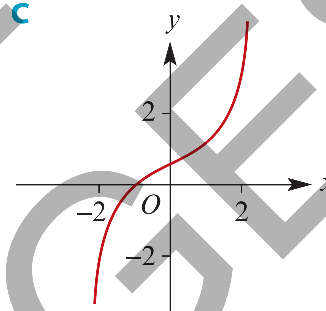

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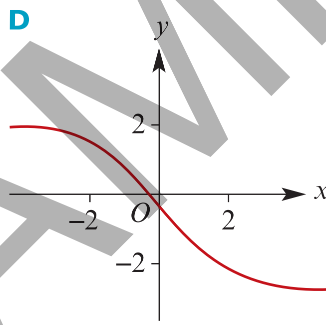

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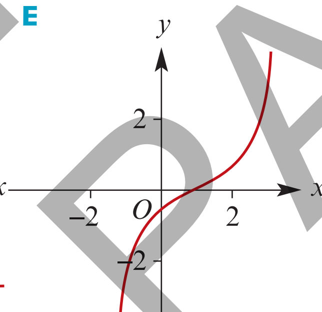


17 Let $f(x) = 3x - 2$ and $g(x) = x^2 - 4x + 2$. A sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$ is

A a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{2}{3}$ units in the positive direction of the x-axis

B a dilation of factor 3 from the y-axis followed by a translation 2 units in the negative direction of the x-axis

C a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{1}{2}$ unit in the positive direction of the x-axis

D a dilation of factor 3 from the y-axis followed by a translation 2 units in the positive direction of the x-axis

E a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 2 units in the positive direction of the x-axis

CF

Extended-response questions

1 A firm manufacturing jackets is capable of producing 100 jackets per day, but it can only sell all of these if the charge to wholesalers is no more than \$50 per jacket. On the other hand, at the current price of \$75 per jacket, only 50 can be sold per day. Assume that the graph of price, \$$P$, against number sold per day, N, is a straight line.

a Sketch the graph of P against N.

b Find the equation of the straight line.

c Use the equation to find:

i the price at which 88 jackets per day could be sold

ii the number of jackets that should be manufactured to sell at \$60 each.

2 A new town was built 10 years ago to house the workers of a woollen mill established in a remote country area. Three years after the town was built, it had a population of 12 000 people. Business in the wool trade steadily grew, and eight years after the town was built the population had swelled to 19 240.

a Assuming the population growth can be modelled by a linear relationship, find a suitable relation for the population, p, in terms of t, the number of years since the town was built.

b Sketch the graph of p against t, and interpret the p-axis intercept.

c Find the current population of the town.

d Calculate the average rate of growth of the town.

3 $ABCD$ is a quadrilateral with angle ABC a right angle. The point D lies on the perpendicular bisector of AB. The coordinates of A and B are $(7, 2)$ and $(2, 5)$ respectively. The equation of line AD is $y = 4x - 26$.

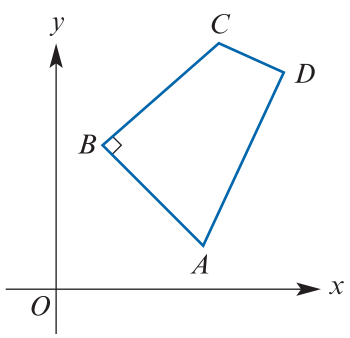
a Find the equation of the perpendicular bisector of line segment AB.

b Find the coordinates of point D.

c Find the gradient of line BC.

d Find the value of the second coordinate c of the point $C(8, c)$.

e Find the area of quadrilateral $ABCD$.



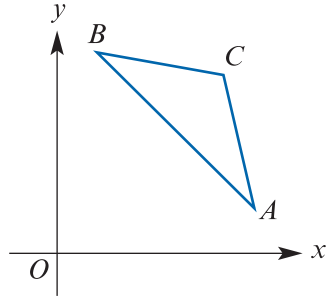
4 Triangle ABC is isosceles with $BC = AC$. The coordinates of the vertices are $A(6, 1)$ and $B(2, 8)$.

a Find the equation of the perpendicular bisector of AB.

b The x-coordinate of C is 3.5. Find the y-coordinate.

c Find the length of AB.

d Find the area of triangle ABC.



Review

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5 If $A = (-4, 6)$ and $B = (6, -7)$, find:

- **a** the coordinates of the midpoint of AB
- **b** the length of AB
- **c** the distance between A and B
- **d** the equation of AB
- **e** the equation of the perpendicular bisector of AB
- **f** the coordinates of the point P on the line segment AB such that $AP : PB = 3 : 1$
- **g** the coordinates of the point P on the line AB such that $AP : AB = 3 : 1$ and P is closer to point B than to point A.

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6 **a** **i** Find the dilation from the x-axis which takes $y = x^2$ to the parabola with its vertex at the origin that passes through the point $(25, 15)$.

ii State the rule which reflects this dilated parabola in the x-axis.

iii State the rule which takes the reflected parabola of part **ii** to a parabola with x-axis intercepts $(0, 0)$ and $(50, 0)$ and vertex $(25, 15)$.

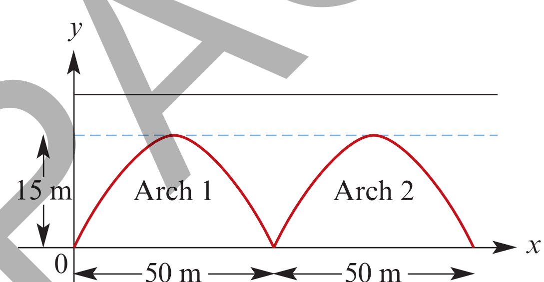
iv State the rule which takes the curve $y = x^2$ to the parabola defined in part **iii**.

b The plans for the entrance of a new building involve twin parabolic arches as shown in the diagram.

i From the results of part **a**, give the equation for the curve of arch 1.

ii Find the translation which maps the curve of arch 1 to the curve of arch 2.

iii Find the equation of the curve of arch 2.



c The architect wishes to have flexibility in her planning and so wants to develop an algorithm for determining the equations of the curves when each arch has width m metres and height n metres.

i Find the rule for the transformation which takes the graph of $y = x^2$ to the current arch 1 with these new dimensions.

ii Find the equation for the curve of arch 1.

iii Find the equation for the curve of arch 2.