1 Functions and relations

Objectives

- To revise **set notation**, including the notation for **sets of numbers**.
- To understand the concepts of **relation** and **function**.
- To find the **domain** and **range** of a given relation.
- To find the **implied (maximal) domain** of a function.
- To work with **restrictions of a function**, **piecewise-defined functions**, **odd functions** and **even functions**.
- To combine functions using **sums**, **products**, **quotients** and **compositions**.
- To understand the concepts of **strictly increasing** and **strictly decreasing**.
- To work with **power functions** and their graphs.
- To apply a knowledge of functions to **solving problems**.

The first five chapters of this book revise and extend some important concepts and techniques from Mathematical Methods Units 1 & 2 that will be built on in Units 3 & 4.

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

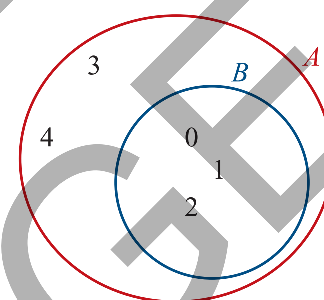
In Chapters 2 to 4 we study different families of functions and their graphs. We revise transformations of the plane in Chapter 2, and then study polynomial functions in Chapter 3 and trigonometric functions in Chapter 4.

1A Set notation and sets of numbers

▶ Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects.
- The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
- If x is **not an element** of A, we write $x \notin A$.
- A set B is called a **subset** of a set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.

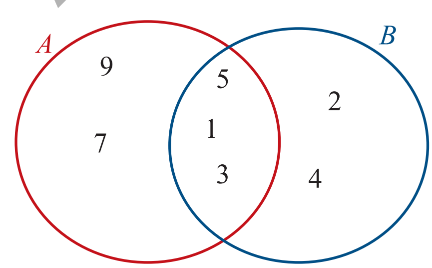


For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then

$$3 \in A, \quad 3 \notin B \quad \text{and} \quad B \subseteq A$$

as illustrated in the Venn diagram opposite.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.



For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$$A \cap B = \{1, 3, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution	*Explanation*
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

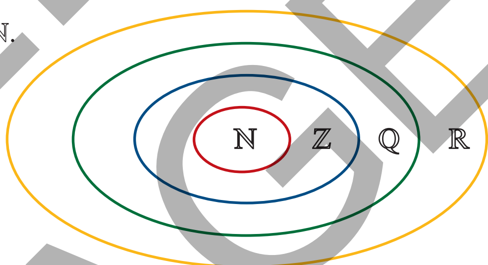
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{ x : \_\_\_ \}$.

For example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ x : x \neq 0 \}$ is the set of all real numbers other than 0.

The following are subsets of the real numbers for which we have special notation:

Positive real numbers $\mathbb{R}^+ = \{ x : x > 0 \}$

Negative real numbers $\mathbb{R}^- = \{ x : x < 0 \}$

Set difference

Sometimes we want to describe a set of real numbers by specifying which numbers are left out. We can do this using **set difference**.

The set $A \setminus B$ contains the elements of A that are not elements of B.

For example:

- $\mathbb{R} \setminus \{0\}$ is the set of all real numbers excluding 0
- $\mathbb{R} \setminus \{1\}$ is the set of all real numbers excluding 1
- $\mathbb{N} \setminus \{5, 7\}$ is the set of all natural numbers excluding 5 and 7.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 2.

Example 2

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

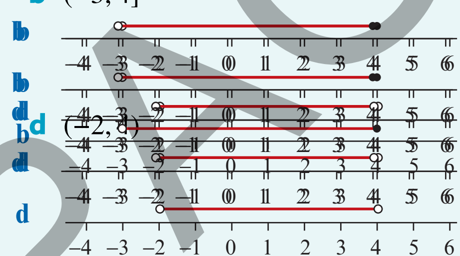
a $[-2, 3]$

c $(-\infty, 5]$

e $(-3, \infty)$

b $(-3, 4]$

d $(-2, 4)$



Note: The 'closed' circle (•) indicates that the number is included.
The 'open' circle (∘) indicates that the number is not included.

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- Intersection The set $A \cap B$ contains the elements in common to A and B.
- Union The set $A \cup B$ contains the elements that are in A or in B (or in both).
- Set difference The set $A \setminus B$ contains the elements of A that are not in B.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.

- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 1A

Example 1

1 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

SF

Example 2

2 Illustrate each of the following intervals on a number line:

a $[-2, 3)$ **b** $(-\infty, 4]$ **c** $[-3, -1]$
d $(-3, \infty)$ **e** $(-4, 3)$ **f** $(-1, 4]$

3 For $X = \{2, 3, 5, 7, 9, 11\}$, $Y = \{7, 9, 15, 19, 23\}$ and $Z = \{2, 7, 9, 15, 19\}$, find:

a $X \cap Y$ **b** $X \cap Y \cap Z$ **c** $X \cup Y$ **d** $X \setminus Y$
e $Z \setminus Y$ **f** $X \cap Z$ **g** $[-2, 8] \cap X$ **h** $(-3, 8] \cap Y$
i $(2, \infty) \cap Y$ **j** $(3, \infty) \cup Y$

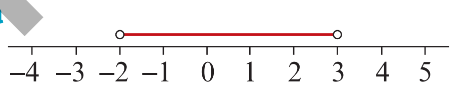
4 For $X = \{a, b, c, d, e\}$ and $Y = \{a, e, i, o, u\}$, find:

a $X \cap Y$ **b** $X \cup Y$ **c** $X \setminus Y$ **d** $Y \setminus X$

5 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{ x : -3 \le x < 1 \}$ **b** $\{ x : -4 < x \le 5 \}$ **c** $\{ y : -\sqrt{2} < y < 0 \}$
d $\left\{ x : -\frac{1}{\sqrt{2}} < x < \sqrt{3} \right\}$ **e** $\{ x : x < -3 \}$ **f** $\mathbb{R}^+$
g $\mathbb{R}^-$ **h** $\{ x : x \ge -2 \}$

6 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.:

a


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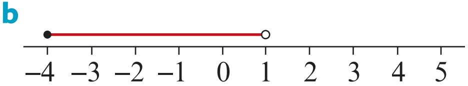
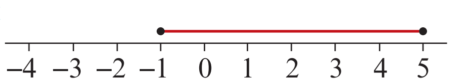
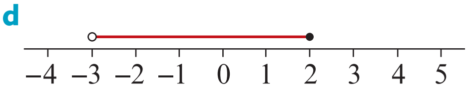


c


d


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7 Illustrate each of the following intervals on a number line:

a $(-3, 2]$ **b** $(-4, 3)$ **c** $(-\infty, 3)$ **d** $[-4, -1]$ **e** $[-4, \infty)$ **f** $[-2, 5)$

8 For each of the following, use one number line on which to represent the sets:

a $[-3, 6],\ [2, 4],\ [-3, 6] \cap [2, 4]$ **b** $[-3, 6],\ \mathbb{R} \setminus [-3, 6]$

c $[-2, \infty),\ (-\infty, 6],\ [-2, \infty) \cap (-\infty, 6]$ **d** $(-8, -2),\ \mathbb{R}^- \setminus (-8, -2)$

1B Identifying and describing relations and functions

▶ Relations, domain and range

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$
b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets:

- The set of all the first coordinates of the ordered pairs is called the **domain**.
- The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$
b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The domain is the set $\{1, 2, 3, 4\}$ and the range is the set $\{2, 3, 4, 5\}$.

Describing relations

Often set notation is not used when describing a relation. For example:

- $\{ (x, y) : y = x^2 \}$ is written as $y = x^2$
- $\{ (x, y) : y = \sqrt{x},\ x \ge 0 \}$ is written as $y = \sqrt{x},\ x \ge 0$.

When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

- $y = x^2$ is assumed to have domain $\mathbb{R}$
- $y = \sqrt{x}$ is assumed to have domain $[0, \infty)$.

Example 3

Sketch the graph of each of the following relations and state the domain and range of each:

a $y = x^2$

b $\{(x, y) : y \leq x + 1\}$

c $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

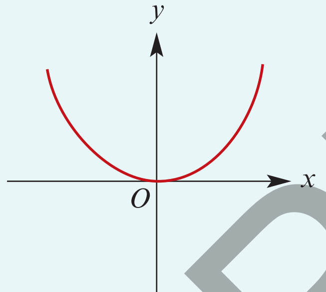
d $x^2 + y^2 = 1$

e $2x + 3y = 6,\ x \geq 0$

f $y = 2x - 1,\ x \in [-1, 2]$

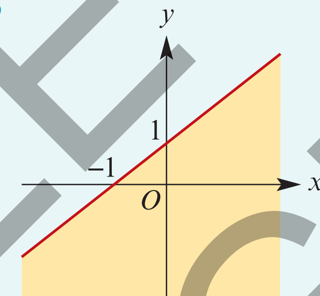
Solution

a



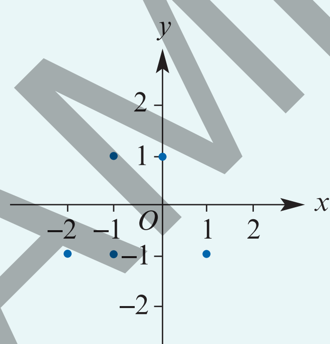
Domain = $\mathbb{R}$; Range = $[0, \infty)$

b



Domain = $\mathbb{R}$; Range = $\mathbb{R}$

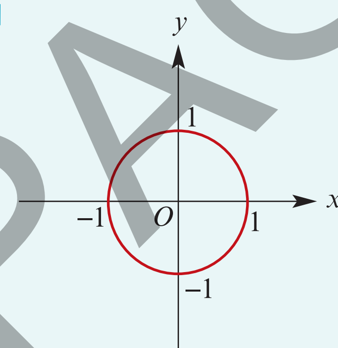
c



Domain = $\{-2, -1, 0, 1\}$

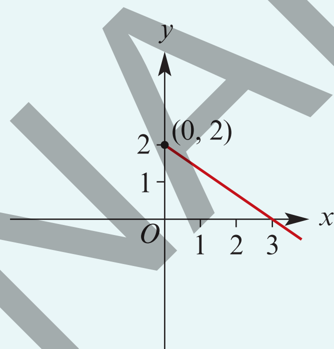
Range = $\{-1, 1\}$

d



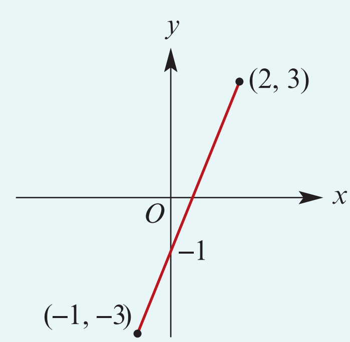
Domain = $[-1, 1]$; Range = $[-1, 1]$

e



Domain = $[0, \infty)$; Range = $(-\infty, 2]$

f



Domain = $[-1, 2]$; Range = $[-3, 3]$

Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 4

Which of the following sets of ordered pairs defines a function?

a $S = \{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ **b** $T = \{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a S is a function, because for each x-value there is only one y-value.

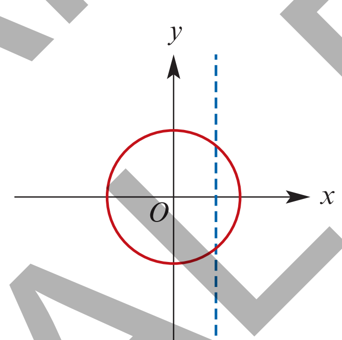
b T is not a function, because there is an x-value with two different y-values: the two ordered pairs $(-4, 1)$ and $(-4, -1)$ in T have the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

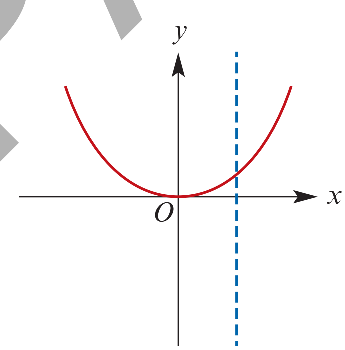
Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function



$y = x^2$ is a function

Functions are usually denoted by lowercase letters such as f, g, h.

If f is a function, then for each x in the domain of f there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

For $(x, y) \in f$, the element y is determined by x, and so we also use the notation $f(x)$, read as 'f of x', in place of y.

For example, instead of $y = 2x + 1$ we can write $f(x) = 2x + 1$. Then $f(5)$ means the y-value obtained when $x = 5$. Therefore $f(5) = 2 \times 5 + 1 = 11$.

By incorporating this notation, we have an alternative way of writing functions:

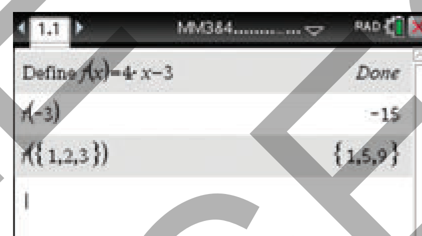
- For the function with rule $y = x^2$ and domain $\mathbb{R}$, we write $f(x) = x^2$, $x \in \mathbb{R}$.
- For the function with rule $y = 2x - 1$ and domain $[0, 4]$, we write $f(x) = 2x - 1$, $x \in [0, 4]$.

If the domain is $\mathbb{R}$, we often just write the rule. For example: $f(x) = x^2$.

With this notation for functions, the domain of f is written as **dom *f*** and range of f as **ran *f***.

Using the TI-Nspire CX non-CAS

- Use (menu) > **Actions** > **Define** to define the function $f(x) = 4x - 3$.
- To find the value of $f(-3)$, type $f(-3)$ followed by (enter).
- To evaluate $f(1)$, $f(2)$ and $f(3)$, type $f(\{1, 2, 3\})$ followed by (enter).



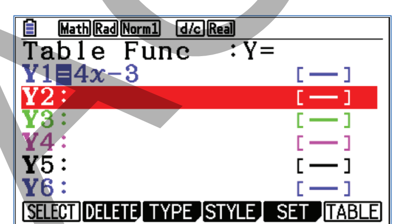
Using the Casio

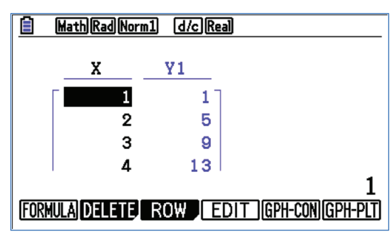
To display a table of values for the function $f(x) = 4x - 3$:

- Press (MENU) (7) to select **Table** mode.
- Enter the rule $y = 4x - 3$ in $Y1$:

  (4) (X,θ,T) (−) (3) (EXE)

- Select **Table** (F6).



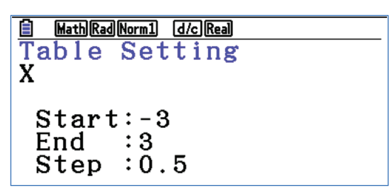


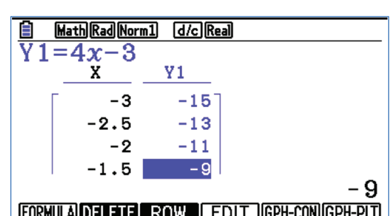
To change the x-values used in the table:

- Press (EXIT) to return to the function list.
- Select **Set** (F5) to adjust the Table Settings. For example, set the x-values to start at -3 and end at 3 with a step size of 0.5:

  ((−)) (3) (EXE) (3) (EXE) (0) (•) (5) (EXE) (EXIT)

- Select **Table** (F6). Use the cursor keys to move around the table.





Example 5

For $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x-1)$

Solution

a $f(3) = 2(3)^2 + 3$
$= 21$

b $f(-2) = 2(-2)^2 - 2$
$= 6$

c $f(x-1) = 2(x-1)^2 + (x-1)$
$= 2(x^2 - 2x + 1) + x - 1$
$= 2x^2 - 3x + 1$

Example 6

For $g(x) = 3x^2 + 1$:

a Find $g(-2)$ and $g(4)$.

b Express each the following in terms of x:

i $g(-2x)$ **ii** $g(x-2)$ **iii** $g(x+2)$ **iv** $g(x^2)$

Solution

a $g(-2) = 3(-2)^2 + 1 = 13$ and $g(4) = 3(4)^2 + 1 = 49$

b **i** $g(-2x) = 3(-2x)^2 + 1$
$= 3 \times 4x^2 + 1$
$= 12x^2 + 1$

ii $g(x-2) = 3(x-2)^2 + 1$
$= 3(x^2 - 4x + 4) + 1$
$= 3x^2 - 12x + 13$

iii $g(x+2) = 3(x+2)^2 + 1$
$= 3(x^2 + 4x + 4) + 1$
$= 3x^2 + 12x + 13$

iv $g(x^2) = 3(x^2)^2 + 1$
$= 3x^4 + 1$

Example 7

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

d Find the pre-image of 6.

Solution

a $f(2) = 2(2) - 4 = 0$
$f(-1) = 2(-1) - 4 = -6$
$f(t) = 2t - 4$

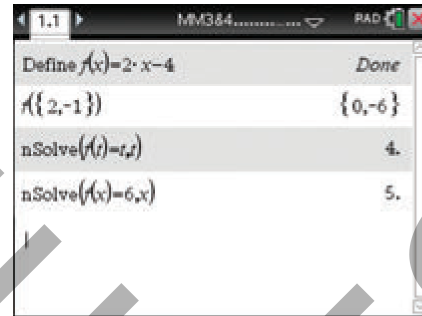
b $f(t) = t$
$2t - 4 = t$
$t - 4 = 0$
$\therefore\ t = 4$

c $f(x) \geq x$
$2x - 4 \geq x$
$x - 4 \geq 0$
$\therefore\ x \geq 4$

d $f(x) = 6$
$2x - 4 = 6$
$x = 5$
Thus 5 is the pre-image of 6.

Using the TI-Nspire CX non-CAS

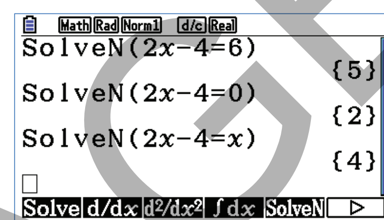
- Use (menu) > **Actions** > **Define** to define the function $f(x) = 2x - 4$.
- Find $f(2)$ and $f(-1)$ as shown.
- Use (menu) > **Algebra** > **Numerical Solve** to solve the equations $f(t) = t$ and $f(x) = 6$.



Using the Casio

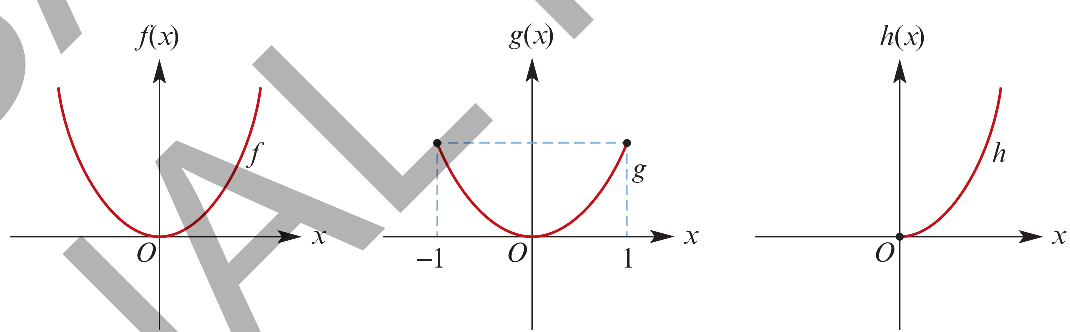
To solve equations involving the function $f(x) = 2x - 4$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** (OPTN) (F4), then **SolveN** (F5).
- To solve $f(x) = 6$, enter the equation $2x - 4 = 6$.
- The equations $f(x) = 0$ and $f(x) = x$ can be solved similarly as shown.



▶ Restriction of a function

Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$ $\qquad g(x) = x^2, \ -1 \le x \le 1$ $\qquad h(x) = x^2, \ x \ge 0$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains.

We call g and h **restrictions** of f, since their domains are subsets of the domain of f.

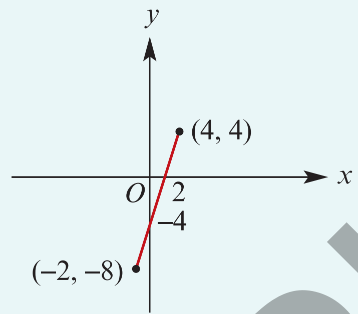
Example 8

For each of the following, sketch the graph and state the range:

a $f(x) = 2x - 4, \ x \in [-2, 4]$ **b** $g(x) = x^2, \ x \in (-1, 2]$

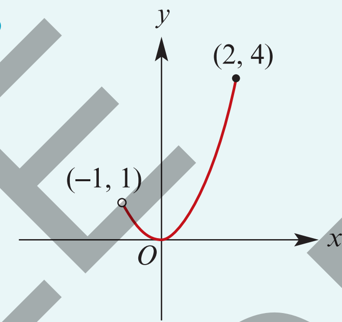
Solution

a



Range = $[-8, 4]$

b



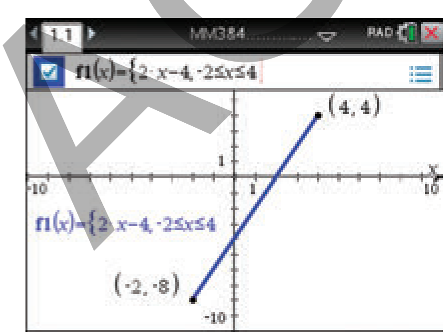
Range = $[0, 4]$

Using the TI-Nspire CX non-CAS

Domain restrictions can be entered with the function if required.

For example: $f1(x) = 2x - 4 \mid -2 \le x \le 4$

Note: The ‘with’ symbol | and the inequality signs can be accessed using (ctrl) (=).



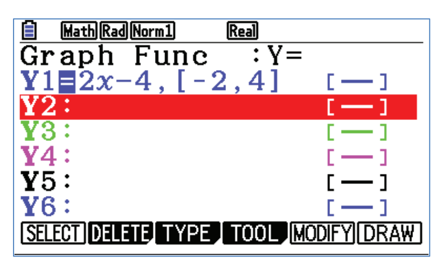
Using the Casio

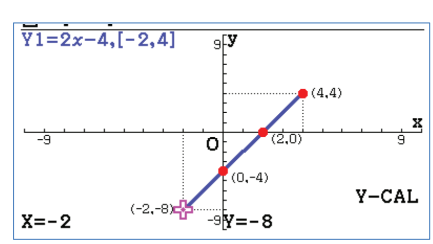
- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = 2x - 4$ and the domain $[-2, 4]$ in $Y1$:

  (2) (X,θ,T) (−) (4) (,)

  (SHIFT) (+) ((−)) (2) (,) (4) (SHIFT) (−) (EXE)
- Select **Draw** (F6) to view the graph. Adjust the View Window (SHIFT) (F3) if required.
- To label key points on the graph, use the **G-Solve** menu (SHIFT) (F5).

Note: When defining a restricted function, always use square brackets to specify the domain (not round brackets).



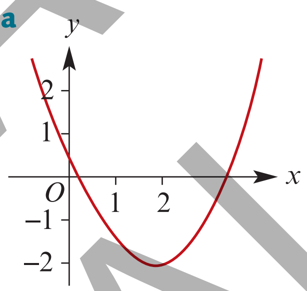


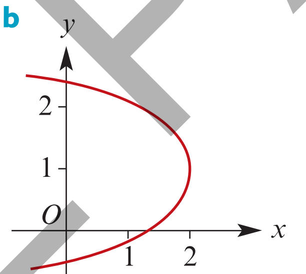
Section summary

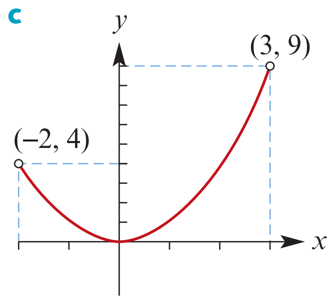
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range: for example, $\{(x, y) : y = x + 1,\ x \geq 0\}$.
- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- Vertical-line test If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y. Since the y-value is determined by the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f(x) = 2x - 1$, $x \in [0, 4]$, to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

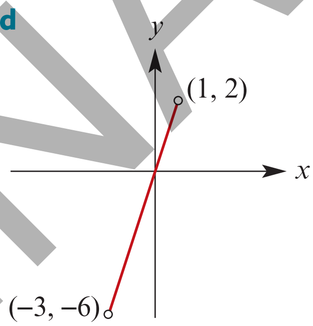
Exercise 1B

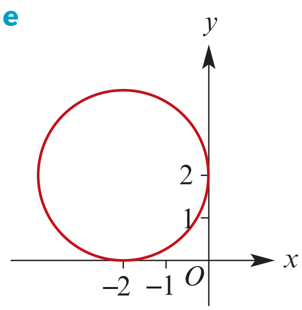
Skillsheet **1** State the domain and range for the relations represented by each of the following graphs:

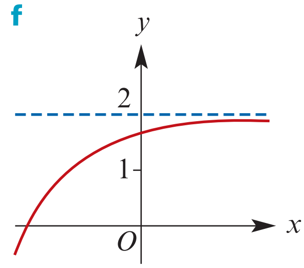
a


b


c


d


e


f


SF

SF

Example 3

2 Sketch a graph of each of the following relations and state its domain and range:

a $y = x^2 + 1$ **b** $x^2 + y^2 = 9$ **c** $3x + 12y = 24, \ x \geq 0$

d $y = \sqrt{2x}$ **e** $y = 5 - x, \ 0 \leq x \leq 5$ **f** $y = x^2 + 2, \ x \in [0, 4]$

g $y = 3x - 2, \ -1 \leq x \leq 2$ **h** $y = 4 - x^2$ **i** $\{(x, y) : y \leq 1 - x\}$

Example 4

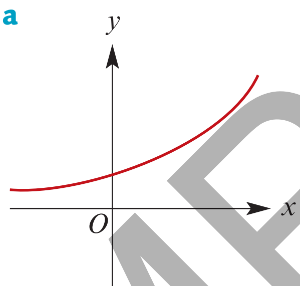
3 Which of the following relations are functions? State the domain and range for each.

a $\{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\}$ **b** $\{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\}$

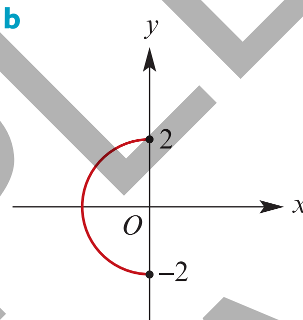
c $\{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\}$ **d** $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\}$

4 Each of the following is the graph of a relation. Which are the graph of a function?

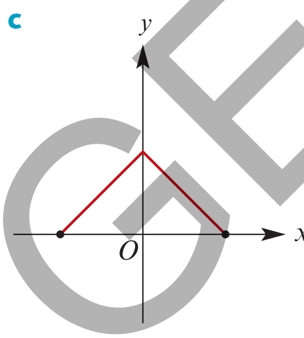
a



b



c



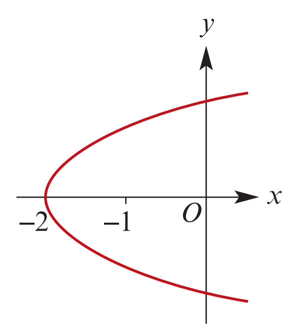
5 Which of the following relations are functions? State the domain and range for each.

a $\{(x, 4) : x \in \mathbb{R}\}$ **b** $\{(2, y) : y \in \mathbb{Z}\}$ **c** $y = -2x + 4$

d $y \geq 3x + 2$ **e** $x^2 + y^2 = 16$

CF

6 The graph of the relation $y^2 = x + 2$ is shown on the right. From this relation, form two functions and specify the range of each.



SF

Example 5

7 Let $f(x) = 2x^2 + 4x$ and $g(x) = 2x^3 + 2x - 6$.

a Evaluate $f(-1)$, $f(2)$, $f(-3)$ and $f(2a)$.

b Evaluate $g(-1)$, $g(2)$, $g(3)$ and $g(a - 1)$.

Example 6

8 Consider the function $g(x) = 3x^2 - 2$.

a Find $g(-2)$ and $g(4)$.

b Express the following in terms of x:

i $g(-2x)$ **ii** $g(x - 2)$ **iii** $g(x + 2)$ **iv** $g(x^2)$

Example 7

9 Consider the function $f(x) = 2x - 3$. Find:

a the image of 3 **b** the pre-image of 11

c the value of x such that $f(x) = 4x$ **d** the values of x such that $f(x) > x$

SF

10 Consider the functions $g(x) = 6x + 7$ and $h(x) = 3x - 2$. Determine the values of x such that:

a $g(x) = h(x)$ **b** $g(x) > h(x)$ **c** $h(x) = 0$

Example 8 **11** Sketch the graph of each of the following and state the range of each:

a $y = x + 1,\ x \geq 2$ **b** $y = -x + 1,\ x \geq 2$
c $y = 2x + 1,\ x \geq -4$ **d** $y = 3x + 2,\ x < 3$
e $y = x + 1,\ x \in (-\infty, 3]$ **f** $y = 3x - 1,\ x \in [-2, 6]$
g $y = -3x - 1,\ x \in [-5, -1]$ **h** $y = 5x - 1,\ x \in (-2, 4)$

12 For $f(x) = 2x^2 - 6x + 1$ and $g(x) = 3 - 2x$:

a Evaluate $f(2)$, $f(-3)$ and $f(-2)$. **b** Evaluate $g(-2)$, $g(1)$ and $g(-3)$.

c Express the following in terms of a:

i $f(a)$ **ii** $f(a + 2)$ **iii** $g(-a)$ **iv** $g(2a)$
v $f(5 - a)$ **vi** $f(2a)$ **vii** $g(a) + f(a)$ **viii** $g(a) - f(a)$

13 For $f(x) = 3x^2 + x - 2$, determine the values of x such that:

a $f(x) = 0$ **b** $f(x) = x$ **c** $f(x) = -2$
d $f(x) > 0$ **e** $f(x) > x$ **f** $f(x) \leq -2$

14 For $f(x) = x^2 + x$, find:

a $f(-2)$ **b** $f(2)$
c $f(-a)$ in terms of a **d** $f(a) + f(-a)$ in terms of a
e $f(a) - f(-a)$ in terms of a **f** $f(a^2)$ in terms of a

15 For $g(x) = 3x - 2$, determine the values of x such that:

a $g(x) = 4$ **b** $g(x) > 4$ **c** $g(x) = a$
d $g(-x) = 6$ **e** $g(2x) = 4$ **f** $\dfrac{1}{g(x)} = 6$

16 Find the value of k for each of the following if $f(3) = 3$, where:

a $f(x) = kx - 1$ **b** $f(x) = x^2 - k$ **c** $f(x) = x^2 + kx + 1$
d $f(x) = \dfrac{k}{x}$ **e** $f(x) = kx^2$ **f** $f(x) = 1 - kx^2$

17 Find the values of x for which the given functions have the given value:

a $f(x) = 5x - 4,\ f(x) = 2$ **b** $f(x) = \dfrac{1}{x},\ f(x) = 5$
c $f(x) = \dfrac{1}{x^2},\ f(x) = 9$ **d** $f(x) = x + \dfrac{1}{x},\ f(x) = 2$
e $f(x) = (x + 1)(x - 2),\ f(x) = 0$

1C Implied domains and types of functions

Implied domains

If the domain of a function is not specified, then the domain is the largest subset of $\mathbb{R}$ for which the rule is defined; this is called the **implied domain** or the **maximal domain**.

Thus, for the function $f(x) = \sqrt{x}$, the implied domain is $[0, \infty)$. We write:

$$f(x) = \sqrt{x}, \ x \in [0, \infty)$$

Example 9

Find the implied domain and the corresponding range for the functions with rules:

a $f(x) = 2x - 3$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{x+6}$ **d** $f(x) = \sqrt{4 - x^2}$

Solution

a $f(x) = 2x - 3$ is defined for all x. The implied domain is $\mathbb{R}$. The range is $\mathbb{R}$.

b $f(x) = \dfrac{1}{(x-2)^2}$ is defined for $x \neq 2$. The implied domain is $\mathbb{R} \setminus \{2\}$. The range is $\mathbb{R}^+$.

c $f(x) = \sqrt{x+6}$ is defined for $x + 6 \geq 0$, i.e. for $x \geq -6$.
Thus the implied domain is $[-6, \infty)$. The range is $[0, \infty)$.

d $f(x) = \sqrt{4 - x^2}$ is defined for $4 - x^2 \geq 0$, i.e. for $x^2 \leq 4$.
Thus the implied domain is $[-2, 2]$. The range is $[0, 2]$.

Example 10

Find the implied domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x-3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x-5} + \sqrt{8-x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is defined when $2x - 3 \neq 0$, i.e. when $x \neq \frac{3}{2}$. Thus the implied domain is $\mathbb{R} \setminus \{\frac{3}{2}\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$. Thus the implied domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$. Thus the implied domain is $[5, 8]$.

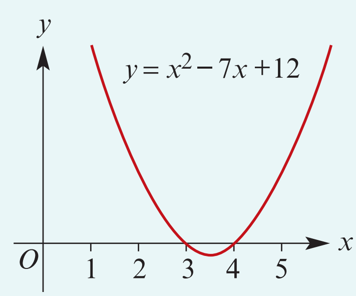
d $f(x)$ is defined when

$$x^2 - 7x + 12 \geq 0$$

which is equivalent to

$$(x-3)(x-4) \geq 0$$

Thus $f(x)$ is defined when $x \geq 4$ or $x \leq 3$.
The implied domain is $(-\infty, 3] \cup [4, \infty)$.



► Piecewise-defined functions

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 11

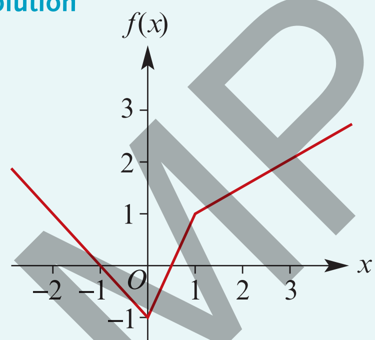
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

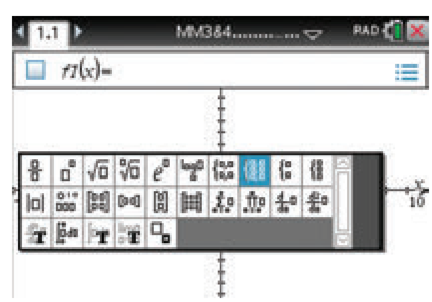
Explanation

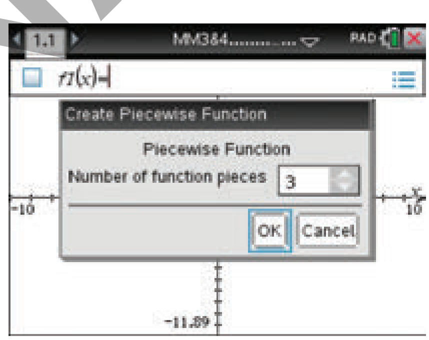
- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$. Note that when $x = 1$, $y = 1$ for this rule.

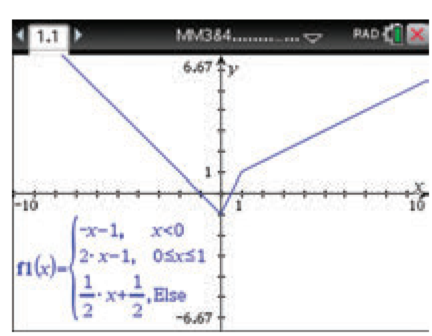
Note: For this function, the sections of the graph 'join up'. This is not always the case.

Using the TI-Nspire CX non-CAS

- In a **Graphs** application with the cursor in the entry line, select the piecewise function template as shown. (Access the template from the 2D-template palette (template key).)
- If the domain of the last function piece is the remaining subset of ℝ, then leave the final condition blank and it will autofill as 'Else' when you press (enter).





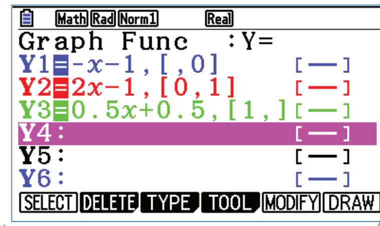


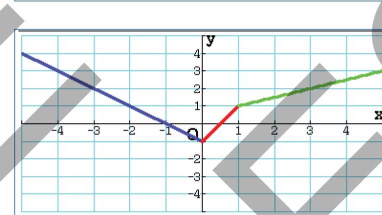
Using the Casio

- Press MENU 5 to select **Graph** mode.
- Enter the first rule, $y = -x - 1$ for $x < 0$, in $Y1$:

  (−) X,θ,T − 1 ,

  SHIFT + , 0 SHIFT − EXE
- Enter the second and third rules in $Y2$ and $Y3$ as shown.
- Select **Draw** F6 to view the graph. Adjust the View Window SHIFT F3 if required.





Note: The syntax for entering a function with a restricted domain is:
function rule, [*start x-value*, *end x-value*]

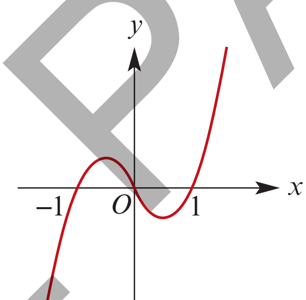
▶ Odd and even functions

Odd functions

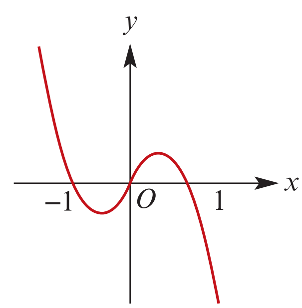
An **odd** function has the property that $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

For example, $f(x) = x^3 - x$ is an odd function, since

$$f(-x) = (-x)^3 - (-x)$$
$$= -x^3 + x$$
$$= -f(x)$$



$y = f(x)$



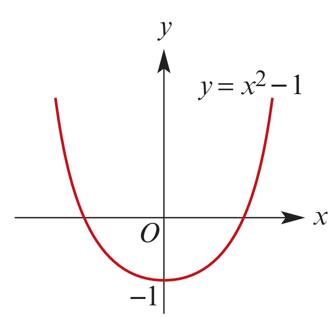
$y = f(-x)$

Even functions

An **even** function has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis.

For example, $f(x) = x^2 - 1$ is an even function, since

$$f(-x) = (-x)^2 - 1$$
$$= x^2 - 1$$
$$= f(x)$$



The properties of odd and even functions often facilitate the sketching of graphs.

Example 12

State whether each function is odd or even or neither:

a $f(x) = x^2 + 7$ **b** $f(x) = x^4 + x^2$ **c** $f(x) = -2x^3 + 7$

d $f(x) = \frac{1}{x}$ **e** $f(x) = \frac{1}{x-3}$ **f** $f(x) = x^5 + x^3 + x$

Solution

a $f(-a) = (-a)^2 + 7$
$= a^2 + 7$
$= f(a)$
The function is even.

b $f(-a) = (-a)^4 + (-a)^2$
$= a^4 + a^2$
$= f(a)$
The function is even.

c $f(-1) = -2(-1)^3 + 7 = 9$
but $f(1) = -2 + 7 = 5$
and $-f(1) = -5$
The function is neither even nor odd.

d $f(-a) = \frac{1}{-a}$
$= -\frac{1}{a}$
$= -f(a)$
The function is odd.

e $f(-1) = -\frac{1}{4}$
but $f(1) = -\frac{1}{2}$
and $-f(1) = \frac{1}{2}$
The function is neither even nor odd.

f $f(-a)$
$= (-a)^5 + (-a)^3 + (-a)$
$= -a^5 - a^3 - a$
$= -f(a)$
The function is odd.

Section summary

- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**.
- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

Exercise 1C

Skillsheet
Example 9

SF

1 State the largest possible domain and range for the functions defined by each of the following rules:

a $y = 4 - x$ **b** $y = \sqrt{x}$ **c** $y = x^2 - 2$ **d** $y = \sqrt{16 - x^2}$

e $y = \frac{1}{x}$ **f** $y = 4 - 3x^2$ **g** $y = \sqrt{x - 3}$

2 Each of the following is the rule of a function. In each case, write down the implied domain and the range.

a $y = 3x + 2$ **b** $y = x^2 - 2$ **c** $f(x) = \sqrt{9 - x^2}$ **d** $g(x) = \frac{1}{x-1}$

Example 10

3 Find the implied domain for each of the following rules:

a $f(x) = \dfrac{1}{x-3}$ **b** $f(x) = \sqrt{x^2 - 3}$ **c** $g(x) = \sqrt{x^2 + 3}$

d $h(x) = \sqrt{x-4} + \sqrt{11-x}$ **e** $f(x) = \dfrac{x^2-1}{x+1}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

g $f(x) = \dfrac{1}{(x+1)(x-2)}$ **h** $h(x) = \sqrt{\dfrac{x-1}{x+2}}$ **i** $f(x) = \sqrt{x - 3x^2}$

j $h(x) = \sqrt{25 - x^2}$ **k** $f(x) = \sqrt{x-3} + \sqrt{12-x}$

Example 11

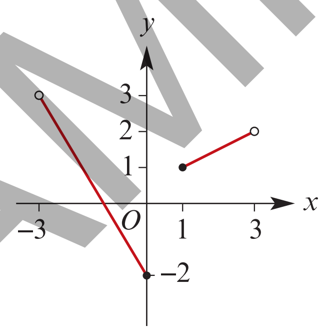
4 **a** Sketch the graph of the function

$$f(x) = \begin{cases} -2x - 2, & x < 0 \\ x - 2, & 0 \le x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$$

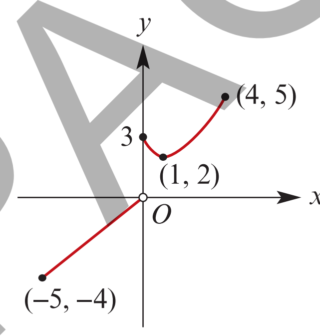
b What is the range of f?

5 For each of the following graphs, state the domain and range of the function:

a



b



6 **a** Sketch the graph of the function

$$f(x) = \begin{cases} 2x + 6, & 0 < x \le 2 \\ -x + 5, & -4 \le x \le 0 \\ -4, & x < -4 \end{cases}$$

b State the domain and range of f.

7 **a** Sketch the graph of the function

$$g(x) = \begin{cases} x^2 + 5, & x > 0 \\ 5 - x, & -3 \le x \le 0 \\ 8, & x < -3 \end{cases}$$

b State the range of g.

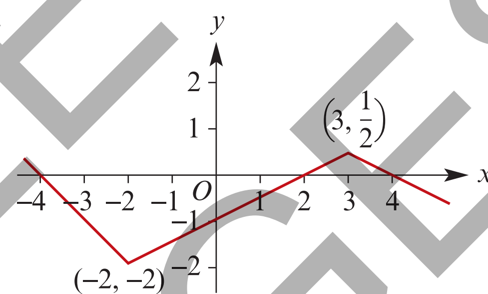
8 Given that

$$f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$$

find:

a $f(-4)$ **b** $f(0)$ **c** $f(4)$
d $f(a + 3)$ in terms of a **e** $f(2a)$ in terms of a **f** $f(a - 3)$ in terms of a

9 Specify the function illustrated by the graph.



Example 12 **10** State whether each of the following functions is odd, even or neither:

a $f(x) = x^4$ **b** $f(x) = x^5$ **c** $f(x) = x^4 - 3x$
d $f(x) = x^4 - 3x^2$ **e** $f(x) = x^5 - 2x^3$ **f** $f(x) = x^4 - 2x^5$

11 State whether each of the following functions is odd, even or neither:

a $f(x) = x^2 - 4$ **b** $f(x) = 2x^4 - x^2$ **c** $f(x) = -4x^3 + 7x$
d $f(x) = \dfrac{1}{2x}$ **e** $f(x) = \dfrac{1}{x + 5}$ **f** $f(x) = 3 + 2x^2$
g $f(x) = x^2 - 5x$ **h** $f(x) = 3^x$ **i** $f(x) = x^4 + x^2 + 2$

1D Combining functions

In this section we look at ways to combine two functions to form a new function, by taking a sum, product, quotient or composition. This enables us to consider quite complicated functions by 'breaking them apart' and viewing them as a combination of simpler functions. We will apply this idea to find the derivatives of functions in Chapters 7 and 8.

▶ Sums and products of functions

The domain of f is denoted by dom f and the domain of g by dom g. Let f and g be functions such that dom $f \cap$ dom $g \neq \varnothing$. The **sum**, $f + g$, and the **product**, fg, as functions on dom $f \cap$ dom g are defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)\,g(x)$$

The domain of both $f + g$ and fg is the intersection of the domains of f and g, i.e. the values of x for which both f and g are defined.

Example 13

If $f(x) = \sqrt{x-2}$ for all $x \ge 2$ and $g(x) = \sqrt{4-x}$ for all $x \le 4$, find:

a $f + g$ **b** $(f+g)(3)$ **c** fg **d** $(fg)(3)$

Solution

Note that dom $f \cap$ dom g $= [2, 4]$.

a $(f+g)(x) = f(x) + g(x)$

$= \sqrt{x-2} + \sqrt{4-x}$

$\text{dom}(f+g) = [2, 4]$

b $(f+g)(3) = \sqrt{3-2} + \sqrt{4-3}$

$= 2$

c $(fg)(x) = f(x)\,g(x)$

$= \sqrt{(x-2)(4-x)}$

$\text{dom}(fg) = [2, 4]$

d $(fg)(3) = \sqrt{(3-2)(4-3)}$

$= 1$

Quotients of functions

Let f and g be functions such that dom $f \cap$ dom $g \cap \{\, x : g(x) \ne 0 \,\} \ne \emptyset$.

The **quotient** $\dfrac{f}{g}$ is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

The domain of the quotient is the intersection of the domains of f and g with the set of all real numbers x for which $g(x) \ne 0$.

Example 14

If $f(x) = x$ for all x and $g(x) = \sqrt{16-x}$ for all $x \le 16$, find:

a $\dfrac{f}{g}$ **b** $\dfrac{f}{g}(7)$ **c** $\dfrac{g}{f}$ **d** $\dfrac{g}{f}(7)$

Solution

Note that dom $f \cap$ dom $g \cap \{\, x : g(x) \ne 0 \,\} = (-\infty, 16)$.

a $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{x}{\sqrt{16-x}}$

$\text{dom}\left(\dfrac{f}{g}\right) = (-\infty, 16)$

b $\dfrac{f}{g}(7) = \dfrac{f(7)}{g(7)} = \dfrac{7}{3}$

Note that dom $g \cap$ dom $f \cap \{\, x : f(x) \ne 0 \,\} = (-\infty, 16] \setminus \{0\}$.

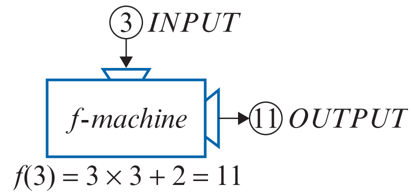
c $\dfrac{g}{f}(x) = \dfrac{g(x)}{f(x)} = \dfrac{\sqrt{16-x}}{x}$

$\text{dom}\left(\dfrac{g}{f}\right) = (-\infty, 16] \setminus \{0\}$

d $\dfrac{g}{f}(7) = \dfrac{g(7)}{f(7)} = \dfrac{3}{7}$

Composition of functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range). For example, the diagram on the right represents an 'f-machine' where $f(x) = 3x + 2$.



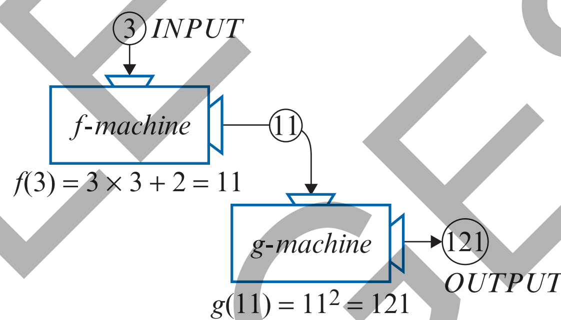
With many processes, more than one machine operation is required to produce an output.

Suppose an output is the result of one function being applied after another.

For example: $f(x) = 3x + 2$

followed by $g(x) = x^2$

This is illustrated on the right.



A new function h is formed. The rule for h is $h(x) = (3x + 2)^2$.

The diagram shows $f(3) = 11$ and then $g(11) = 121$. This may be written:

$$h(3) = g(f(3)) = g(11) = 121$$

The new function h is said to be the **composition** of g with f. This is written $h = g \circ f$ (read 'composition of f followed by g') and the rule for h is given by $h(x) = g(f(x))$.

In the example we have considered:

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^2 \end{aligned}$$

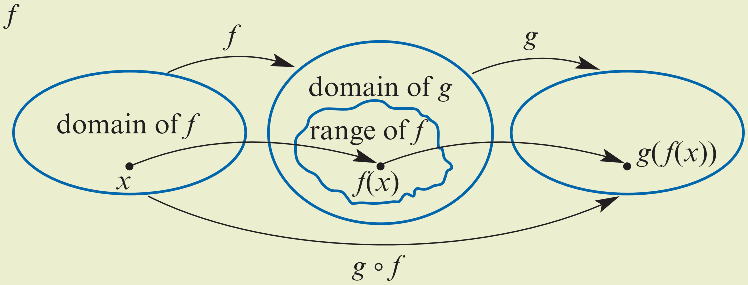
In general, for functions f and g such that

$$\text{ran } f \subseteq \text{dom } g$$

we define the **composite function** of g with f by

$$(g \circ f)(x) = g(f(x))$$

$$\text{dom}(g \circ f) = \text{dom } f$$



Example 15

For the functions $f(x) = 2x - 1$ and $g(x) = 3x^2$, find:

a $f(g(x))$

b $g(f(x))$

c $f(f(x))$

d $g(g(x))$

Solution

a $f(g(x)) = f(3x^2)$
$= 2(3x^2) - 1$
$= 6x^2 - 1$

b $g(f(x)) = g(2x - 1)$
$= 3(2x - 1)^2$
$= 12x^2 - 12x + 3$

c $f(f(x)) = f(2x - 1)$
$= 2(2x - 1) - 1$
$= 4x - 3$

d $g(g(x)) = g(3x^2)$
$= 3(3x^2)^2$
$= 27x^4$

Note: It can be seen from this example that in general $f(g(x)) \neq g(f(x))$.

Example 16

Express each of the following as the composition of two functions:

a $h(x) = (2x + 1)^2$

b $h(x) = \sqrt{x^2}$

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Solution

a $h(x) = (2x + 1)^2$
Choose $f(x) = 2x + 1$
and $g(x) = x^2$.
Then $h(x) = g(f(x))$.

b $h(x) = \sqrt{x^2}$
Choose $f(x) = x^2$
and $g(x) = \sqrt{x}$.
Then $h(x) = g(f(x))$.

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$
Choose $f(x) = x^2 - 2$
and $g(x) = x^n$.
Then $h(x) = g(f(x))$.

Note: These are not the only possible answers, but the 'natural' choices have been made.

Section summary

Combining functions

	Rule	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
Quotient	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$	$\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}\, f \cap \text{dom}\, g \cap \{ x : g(x) \neq 0 \}$
Composition	$(g \circ f)(x) = g(f(x))$	$\text{dom}(g \circ f) = \text{dom}\, f$ if $\text{ran}\, f \subseteq \text{dom}\, g$

Exercise 1D

Skillsheet

SF

1 For each of the following, find $(f+g)(x)$ and $(fg)(x)$ and state the domain for both $f+g$ and fg:

Example 13

a $f(x) = 3x$ and $g(x) = x + 2$

b $f(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $g(x) = x^2$ for all $x \in \mathbb{R}^+$

c $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [1, \infty)$

d $f(x) = x^2$, $x \geq 0$, and $g(x) = \sqrt{4-x}$, $0 \leq x \leq 4$

Example 14

2 For each pair of functions f and g from Question 1:

i find $\dfrac{f}{g}(x)$ and state the domain of $\dfrac{f}{g}$

ii find $\dfrac{g}{f}(x)$ and state the domain of $\dfrac{g}{f}$.

3 Functions f, g, h and k are defined by:

i $f(x) = x^2 + 1, \ x \in \mathbb{R}$

ii $g(x) = x, \ x \in \mathbb{R}$

iii $h(x) = \dfrac{1}{x^2}, \ x \neq 0$

iv $k(x) = \dfrac{1}{x}, \ x \neq 0$

a State which of the above functions are odd and which are even.

b Give the rules for the functions $f+h$, fh, $g+k$, gk, $f+g$ and fg, stating which are odd and which are even.

Example 15

4 For each of the following, find $f(g(x))$ and $g(f(x))$:

a $f(x) = 2x - 1, \ g(x) = 2x$

b $f(x) = 4x + 1, \ g(x) = 2x + 1$

c $f(x) = 2x - 1, \ g(x) = 2x - 3$

d $f(x) = 2x - 1, \ g(x) = x^2$

e $f(x) = 2x^2 + 1, \ g(x) = x - 5$

f $f(x) = 2x + 1, \ g(x) = x^2$

5 For the functions $f(x) = 2x - 1$ and $h(x) = 3x + 2$, find:

a $f(h(x))$ **b** $h(f(x))$ **c** $f(h(2))$ **d** $h(f(2))$

e $f(h(3))$ **f** $h(f(-1))$ **g** $f(h(0))$

6 For the functions $f(x) = x^2 + 2x$ and $h(x) = 3x + 1$, find:

a $f(h(x))$ **b** $h(f(x))$ **c** $f(h(3))$ **d** $h(f(3))$

e $f(h(0))$ **f** $h(f(0))$

Example 16

7 Express each of the following as the composition of two functions:

a $h(x) = (x^2 - 1)^4$

b $h(x) = \sqrt{x^4 + 3}$

c $h(x) = (x^2 - 2x)^n$ where $n \in \mathbb{N}$

d $h(x) = \dfrac{1}{2x + 3}$

e $h(x) = (x^2 - 2x)^3 - 2(x^2 - 2x)$

f $h(x) = 2(2x^2 + 1)^2 + 1$

1E Power functions

We now consider functions with rules of the form $f(x) = x^r$, where r is a rational number. These functions are called **power functions**.

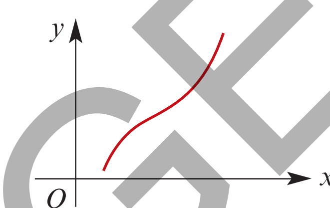
In this section, we look at power functions with rules such as

$$f(x) = x^4, \quad f(x) = x^{-4}, \quad f(x) = x^{\frac{1}{4}}, \quad f(x) = x^5, \quad f(x) = x^{-5}, \quad f(x) = x^{\frac{1}{3}}$$

You may like to investigate further by using your calculator to plot the graphs of more complicated power functions with rules such as $f(x) = x^{\frac{2}{3}}$ and $f(x) = x^{\frac{3}{2}}$.

Increasing and decreasing functions

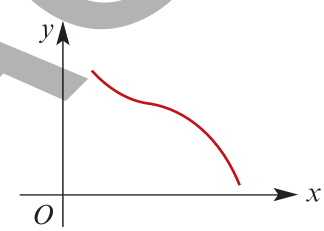
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f(x) = x^2$, $x > 0$, is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

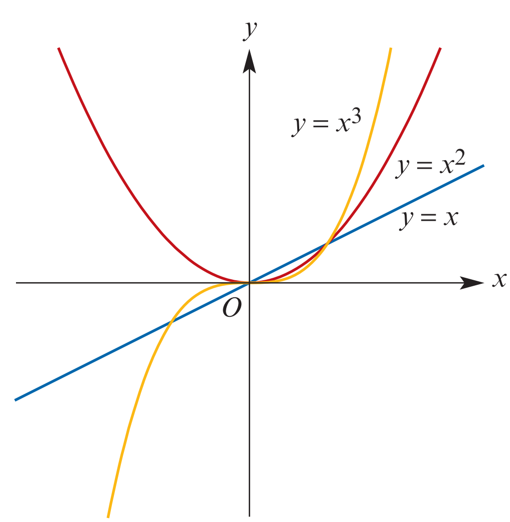


For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f(x) = x^2$, $x < 0$, is strictly decreasing.

Power functions with positive integer index

We start by considering power functions $f(x) = x^n$ where n is a positive integer.



Taking $n = 1, 2, 3$, we obtain the linear function $f(x) = x$, the quadratic function $f(x) = x^2$ and the cubic function $f(x) = x^3$.

We have studied these functions in Mathematical Methods Units 1 & 2 and have referred to them in the earlier sections of this chapter.

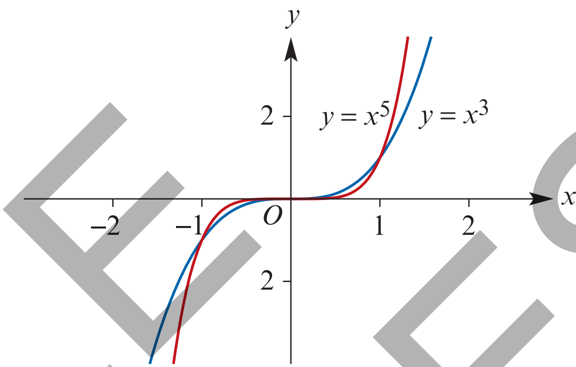
The general shape of the graph of $f(x) = x^n$ depends on whether the index n is odd or even.

The function $f(x) = x^n$ where n is an odd positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

Some properties of $f(x) = x^n$ where n is an odd positive integer:

- f is an odd function
- f is strictly increasing
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$
- as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.

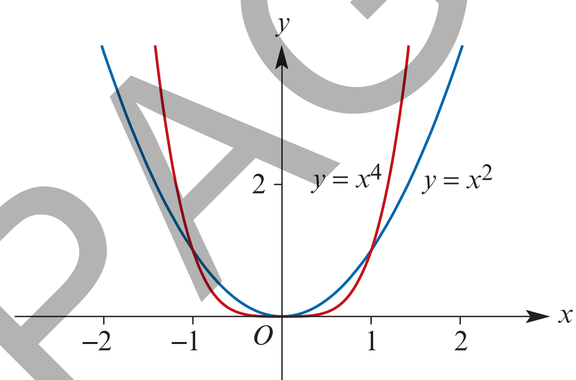


The function $f(x) = x^n$ where n is an even positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $[0, \infty)$.

Some properties of $f(x) = x^n$ where n is an even positive integer:

- f is an even function
- f is strictly increasing for $x > 0$
- f is strictly decreasing for $x < 0$
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = 1$
- as $x \to \pm\infty$, $f(x) \to \infty$.



Power functions with negative integer index

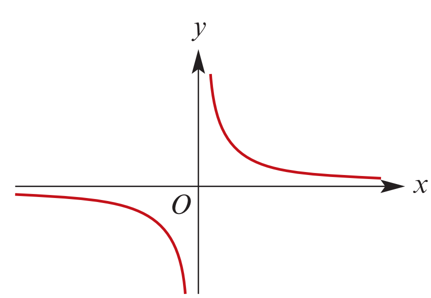
Again, the general shape of the graph depends on whether the index n is odd or even.

The function $f(x) = x^n$ where n is an odd negative integer

Taking $n = -1$, we obtain

$$f(x) = x^{-1} = \frac{1}{x}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the function $f(x) = x^{-k}$, where $k = 1, 3, 5, \ldots$

- the maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$
- f is an odd function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

Example 17

For the function f with rule $f(x) = \frac{1}{x^5}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

Solution

a The maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$.

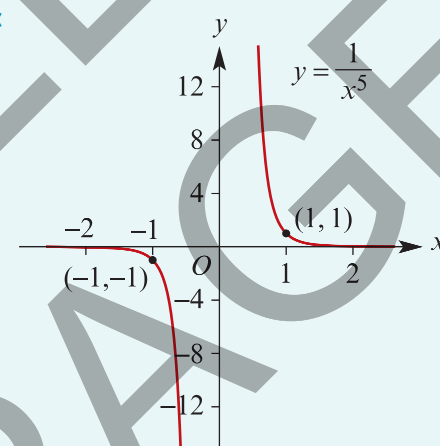
b **i** $f(2) = \frac{1}{2^5} = \frac{1}{32}$

ii $f(-2) = \frac{1}{(-2)^5} = -\frac{1}{32}$

iii $f(\frac{1}{2}) = \frac{1}{(\frac{1}{2})^5} = 32$

iv $f(-\frac{1}{2}) = \frac{1}{(-\frac{1}{2})^5} = -32$

c



Example 18

Let $f(x) = x^{-1}$ for $x \in \mathbb{R} \setminus \{0\}$ and $g(x) = x^{-3}$ for $x \in \mathbb{R} \setminus \{0\}$.

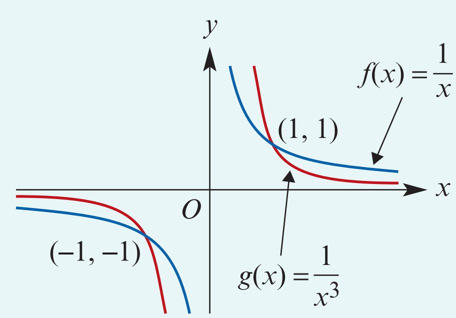
a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a
$$f(x) = g(x)$$
$$x^{-1} = x^{-3}$$
$$\frac{1}{x} = \frac{1}{x^3}$$
$$x^2 = 1$$
$$\therefore \quad x = 1 \text{ or } x = -1$$

b



Note:

If $x > 1$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

If $0 < x < 1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

If $x < -1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

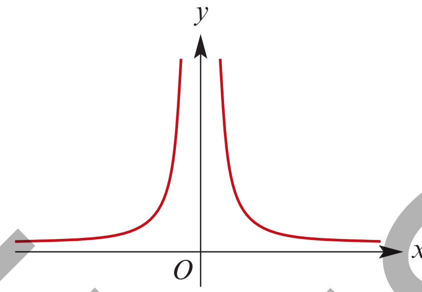
If $-1 < x < 0$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

The function $f(x) = x^n$ where n is an even negative integer

Taking $n = -2$, we obtain

$$f(x) = x^{-2} = \frac{1}{x^2}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the function $f(x) = x^{-k}$, where $k = 2, 4, 6, \dots$

- the maximal domain $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R}^+$
- f is an even function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

▶ The function $f(x) = x^{\frac{1}{n}}$ where n is a positive integer

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

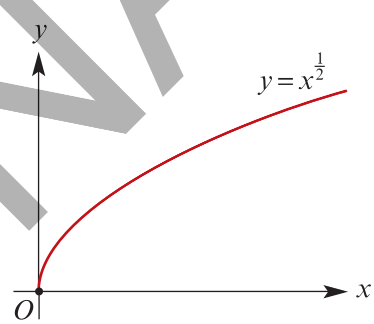
If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

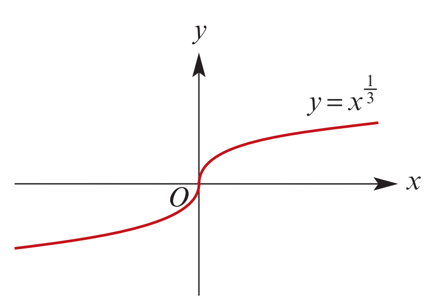
In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

Let $f(x) = x^{\frac{1}{n}}$. When n is even the maximal domain is $[0, \infty)$ and when n is odd the maximal domain is $\mathbb{R}$. The graphs of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ are as shown.





Example 19

Consider the functions $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{1}{2}}$, $x \geq 0$.

a Find the values of x for which $f(x) = g(x)$.

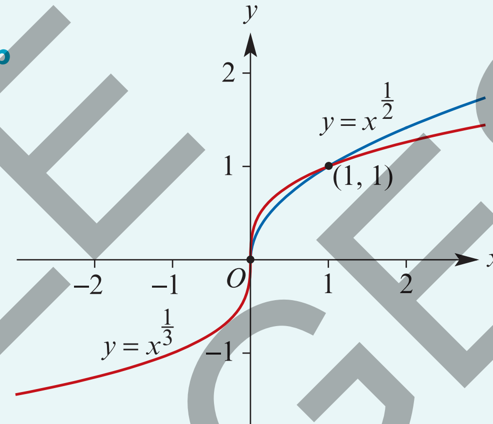
b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a

$$f(x) = g(x)$$
$$x^{\frac{1}{3}} = x^{\frac{1}{2}}$$
$$x^{\frac{1}{3}} - x^{\frac{1}{2}} = 0$$
$$x^{\frac{1}{3}}\left(1 - x^{\frac{1}{6}}\right) = 0$$
$$\therefore \quad x = 0 \text{ or } 1 - x^{\frac{1}{6}} = 0$$
$$\therefore \quad x = 0 \text{ or } x = 1$$

b

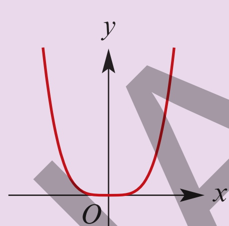


Section summary

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

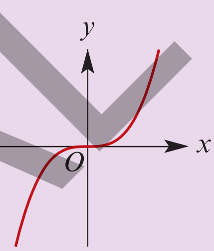
Even positive

$f(x) = x^4$



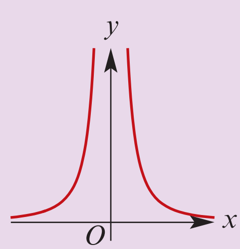
Odd positive

$f(x) = x^3$



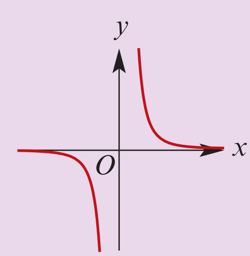
Even negative

$f(x) = x^{-2}$



Odd negative

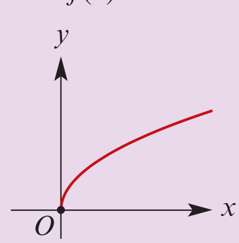
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

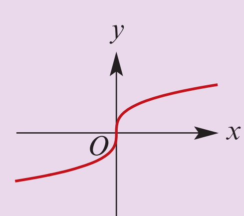
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Exercise 1E

Example 17 **1** For the function f with rule $f(x) = \frac{1}{x^4}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

2 For each of the following, state whether the function is odd, even or neither:

a $f(x) = 2x^5$ **b** $f(x) = x^2 + 3$ **c** $f(x) = x^{\frac{1}{5}}$

d $f(x) = \frac{1}{x}$ **e** $f(x) = \frac{1}{x^2}$ **f** $f(x) = \sqrt[3]{x}$

Example 18 **3** Let $f(x) = x^{-2}$, $x \neq 0$, and $g(x) = x^{-4}$, $x \neq 0$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 19 **4** Let $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{1}{4}}$, $x \geq 0$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

1F Applications of functions

In this section we use function notation in the solution of some problems.

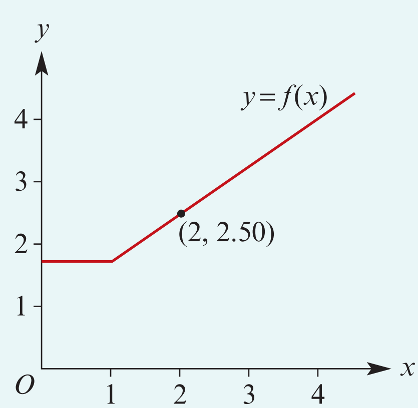
Example 20

The cost of a taxi trip in a particular city is \$1.75 up to and including 1 km. After 1 km the passenger pays an additional 75 cents per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$.

Solution

Let x denote the length of the trip in kilometres.
Then the cost in dollars is given by

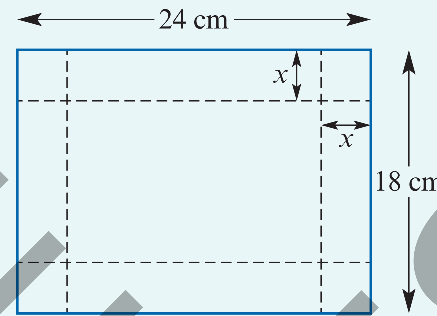
$$f(x) = \begin{cases} 1.75 & \text{for } 0 \leq x \leq 1 \\ 1.75 + 0.75(x-1) & \text{for } x > 1 \end{cases}$$



Example 21

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function V which gives the volume of the box in terms of x, and state the domain of the function.



Solution

The dimensions of the box will be $24 - 2x$, $18 - 2x$ and x.

Thus the volume of the box is determined by the function

$$V(x) = (24 - 2x)(18 - 2x)x$$

For the box to be formed:

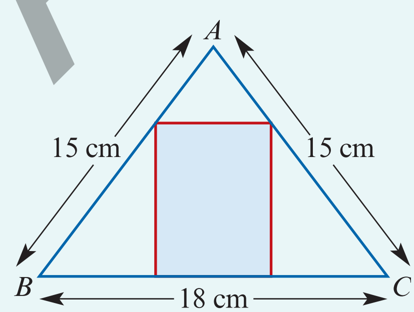
$$24 - 2x \geq 0 \quad \text{and} \quad 18 - 2x \geq 0 \quad \text{and} \quad x \geq 0$$

Therefore $x \leq 12$ and $x \leq 9$ and $x \geq 0$. The domain of V is $[0, 9]$.

Example 22

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area-of-the-rectangle function and state the domain.

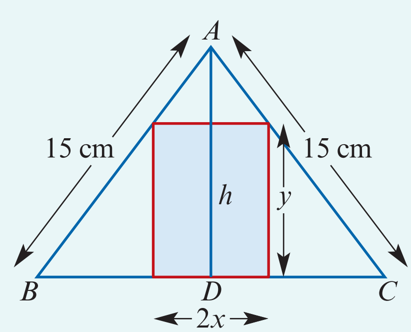


Solution

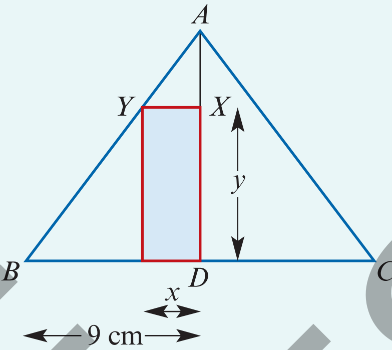
Let the height of the rectangle be y cm and the width $2x$ cm.

The height (h cm) of the triangle can be determined by Pythagoras' theorem:

$$h = \sqrt{15^2 - 9^2} = 12$$



In the diagram opposite, the triangle AYX is similar to the triangle ABD. Therefore



$$\frac{x}{9} = \frac{12 - y}{12}$$

$$\frac{12x}{9} = 12 - y$$

$$\therefore \quad y = 12 - \frac{12x}{9}$$

The area of the rectangle is $A = 2xy$, and so

$$A(x) = 2x\left(12 - \frac{12x}{9}\right)$$

$$= \frac{24x}{9}(9 - x)$$

For the rectangle to be formed, we need

$$x \geq 0 \quad \text{and} \quad 12 - \frac{12x}{9} \geq 0$$

$$\therefore \quad x \geq 0 \quad \text{and} \quad x \leq 9$$

The domain is $[0, 9]$, and so the function is given by $A(x) = \frac{24x}{9}(9 - x), \ x \in [0, 9]$.

Exercise 1F

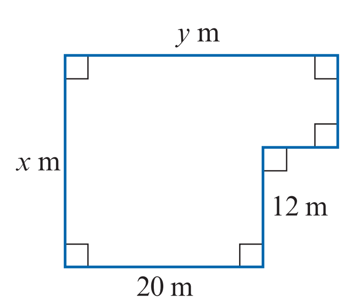
Example 20

1 The cost of a taxi trip in a particular city is $4.00 up to and including 2 km. After 2 km the passenger pays an additional $2.00 per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$, where x is the number of kilometres travelled. (Use a continuous model.)

Example 21

2 A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps. Find a function V which gives the volume of the box in terms of x, and state the domain for the function.

3 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.



a Find a rule for the area, A m^2, of the enclosure in terms of x.

b State a suitable domain of the function $A(x)$.

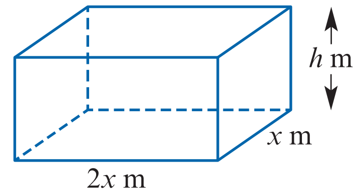
c Sketch the graph of A against x.

d Find the maximum possible area of the enclosure and state the corresponding values of x and y.

CF

CF

4 A cuboid tank is open at the top and the internal dimensions of its base are x m and $2x$ m. The height is h m. The volume of the tank is V m^3 and the volume is fixed. Let S m^2 denote the internal surface area of the tank.



a Find S in terms of:

i x and h

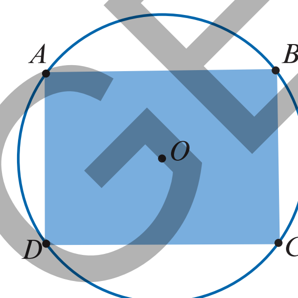
ii V and x

b State the maximal domain for the function defined by the rule in part **a ii**.

c If $2 \le x \le 15$, find the maximum value of S if $V = 1000$ m^3.

Example 22

5 A rectangle $ABCD$ is inscribed in a circle of radius a. Find an area-of-the-rectangle function and state the domain.



CU

6 Let $f(x) = \dfrac{6}{x+2}$ for $x \in [0, 6]$.

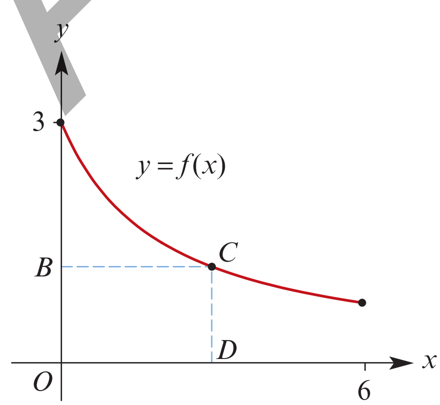
Rectangle $OBCD$ is formed so that the coordinates of C are $(a, f(a))$.

a Find an expression for the area-of-rectangle function A.

b State the implied domain and range of A.

c State the maximum value of $A(x)$ for $x \in [0, 6]$.

d Sketch the graph of $y = A(x)$ for $x \in [0, 6]$.



7 A man walks at a speed of 2 km/h for 45 minutes and then runs at 4 km/h for 30 minutes. Let S km be the distance the man has travelled after t minutes. The distance travelled can be described by

$$S(t) = \begin{cases} at & \text{if } 0 \le t \le c \\ bt + d & \text{if } c < t \le e \end{cases}$$

a Find the values a, b, c, d, e.

b Sketch the graph of $S(t)$ against t.

c State the range of the function.

Chapter summary

AS Nrich

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- A **function** is a relation such that no two ordered pairs in the relation have the same first coordinate.
- For each x in the domain of a function f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **value** of f at x and is denoted by $f(x)$.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- For a function f, the domain is denoted by **dom** $\boldsymbol{f}$ and the range by **ran** $\boldsymbol{f}$.

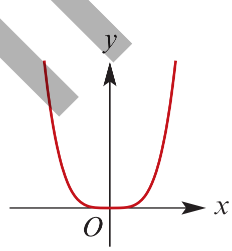
Combining functions

	Rule	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
Quotient	$\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)}$	$\text{dom}\left(\dfrac{f}{g}\right) = \text{dom}\, f \cap \text{dom}\, g \cap \{\, x : g(x) \neq 0 \,\}$
Composition	$(g \circ f)(x) = g(f(x))$	$\text{dom}(g \circ f) = \text{dom}\, f$ if $\text{ran}\, f \subseteq \text{dom}\, g$

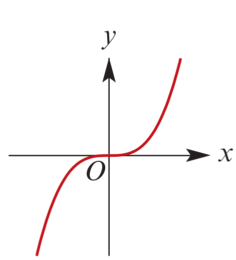
Types of functions

- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

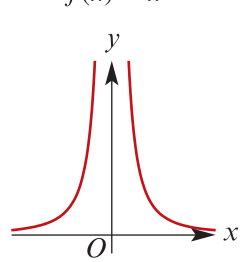
Even positive
$f(x) = x^4$



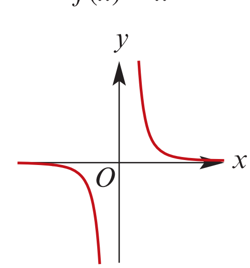
Odd positive
$f(x) = x^3$



Even negative
$f(x) = x^{-2}$



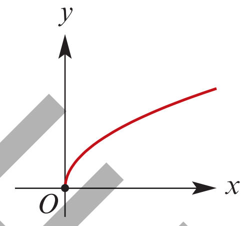
Odd negative
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

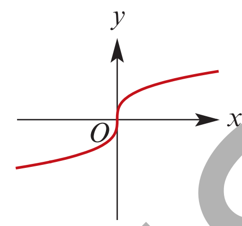
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Technology-free questions

1 Sketch the graph of each of the following relations and state the implied domain and range:

a $f(x) = x^2 + 1$ **b** $f(x) = 2x - 6$ **c** $\{(x, y) : x^2 + y^2 = 25\}$

d $\{(x, y) : y \geq 2x + 1\}$ **e** $\{(x, y) : y < x - 3\}$

2 Consider the function defined by $g(x) = \dfrac{x + 3}{2}$ for $x \in [0, 5]$.

a Sketch the graph of $y = g(x)$.

b State the range of g.

c Determine the value of x such that $g(x) = 4$.

3 For $g(x) = 5x + 1$, find the values of x such that:

a $g(x) = 2$ **b** $g(x) \leq 2$ **c** $\dfrac{1}{g(x)} = 2$

4 Sketch the graph of the function f given by

$$f(x) = \begin{cases} x + 1 & \text{for } x > 2 \\ x^2 - 1 & \text{for } 0 \leq x \leq 2 \\ -x^2 & \text{for } x < 0 \end{cases}$$

5 Find the implied domain for each of the following:

a $f(x) = \dfrac{1}{2x - 6}$ **b** $g(x) = \dfrac{1}{\sqrt{x^2 - 5}}$

c $h(x) = \dfrac{1}{(x - 1)(x + 2)}$ **d** $h(x) = \sqrt{25 - x^2}$

e $f(x) = \sqrt{x - 5} + \sqrt{15 - x}$ **f** $h(x) = \dfrac{1}{3x - 6}$

6 For $f(x) = (x + 2)^2$ and $g(x) = x - 3$, find $(f + g)(x)$ and $(fg)(x)$.

7 Let $f(x) = (x - 1)^2$ for $x \in [1, 5]$ and $g(x) = 2x$ for $x \in \mathbb{R}$. Find $f + g$ and fg.

8 For $f(x) = 2x + 3$ and $g(x) = -x^2$, find:

a $(f + g)(x)$ **b** $(fg)(x)$ **c** the values of x such that $(f + g)(x) = 0$

9 For $f(x) = 2x + 3$ and $g(x) = -x^3$, find:

a $f(g(x))$ **b** $g(f(x))$ **c** $g(g(x))$ **d** $f(f(x))$

10 Express each of the following as the composition of two functions:

a $h(x) = (x^3 - 1)^{\frac{1}{3}}$ **b** $h(x) = \dfrac{1}{x^2 + 1}$ **c** $h(x) = \dfrac{1}{x^2 - 1}$

Multiple-choice questions

1 The maximal domain of the function with rule $f(x) = \sqrt{6 - 2x}$ is

A $(-\infty, 6]$ **B** $[3, \infty)$ **C** $(-\infty, 6]$ **D** $(3, \infty)$ **E** $(-\infty, 3]$

2 Let $f(x) = -x^2$ for $x \in [-1, 3)$. The range of f is

A $\mathbb{R}$ **B** $(-9, 0]$ **C** $(-\infty, 0]$ **D** $(-9, -1]$ **E** $[-9, 0]$

3 If $f(x) = 3x^2 + 2x$, then $f(2a) =$

A $20a^2 + 4a$ **B** $6a^2 + 2a$ **C** $6a^2 + 4a$ **D** $36a^2 + 4a$ **E** $12a^2 + 4a$

4 Let $f(x) = 10 - x$ for $x \in (a, b]$. The range of f is

A $(10 - a, 10 - b)$ **B** $(10 - a, 10 - b]$ **C** $(10 - b, 10 - a)$
D $(10 - b, 10 - a]$ **E** $[10 - b, 10 - a)$

5 For the function with rule

$$f(x) = \begin{cases} x^2 + 5 & x \geq 3 \\ -x + 6 & x < 3 \end{cases}$$

the value of $f(a + 3)$, where a is a negative real number, is

A $a^2 + 6a + 14$ **B** $-a + 9$ **C** $-a + 3$ **D** $a^2 + 14$ **E** $a^2 + 8a + 8$

6 The range of the function with rule $f(x) = x^2 + 2x - 6$ for $x \in [-2, 4)$ is

A $\mathbb{R}$ **B** $(-3, 18]$ **C** $(-6, 18)$ **D** $[0, 6]$ **E** $[-7, 18)$

7 The maximal domain and range of $f(x) = \dfrac{2x + 1}{x - 1}$ are

A $\mathbb{R} \setminus \{0\}$, $\mathbb{R} \setminus \{2\}$ **B** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{-2\}$ **C** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{2\}$
D $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{1\}$ **E** $\mathbb{R} \setminus \{-2\}$, $\mathbb{R} \setminus \{-1\}$

8 If $f(x) = 3x^2$ and $g(x) = 2x + 1$, then $f(g(a))$ is equal to

A $12a^2 + 3$ **B** $12a^2 + 12a + 3$ **C** $6a^2 + 1$
D $6a^2 + 4$ **E** $4a^2 + 4a + 1$

9 Let $f(x) = \sqrt{x+1}$, $x > -1$, and $g(x) = \sqrt{4-x}$, $x \le 4$. The domain of $f + g$ is

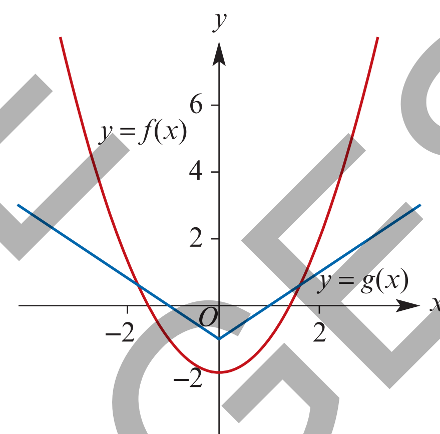
A $\mathbb{R}$ **B** $(-\infty, -1)$ **C** $(-1, 4]$ **D** $(-1, \infty)$ **E** $[-4, 1)$

10 Let $f(x) = 5 - x$ for $x \in D$. If the range of f is $[-2, 3)$, then the domain D is

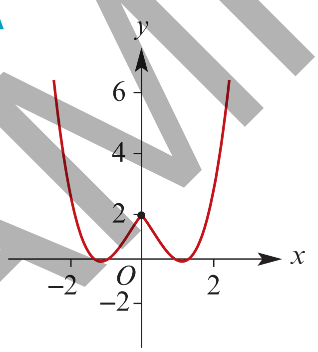
A $[-7, 2)$ **B** $(2, 7]$ **C** $\mathbb{R}$ **D** $[-2, 7)$ **E** $[2, 7)$

11 The graphs of $y = f(x)$ and $y = g(x)$ are as shown on the right.

Which one of the following best represents the graph of $y = f(g(x))$?



A



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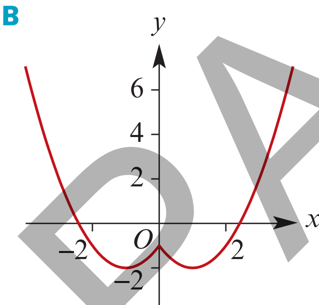

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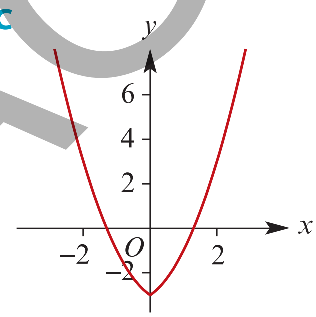

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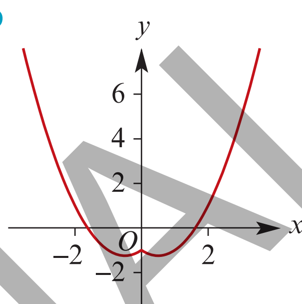

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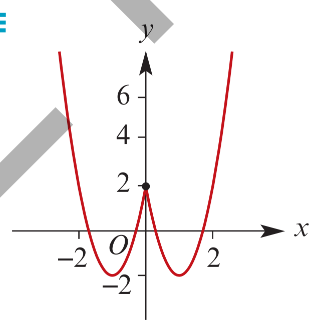


12 Which of the following functions is strictly increasing on the interval $(-\infty, -1]$?

A $f(x) = x^2$ **B** $f(x) = x^4$ **C** $f(x) = x^{\frac{1}{5}}$

D $f(x) = \sqrt{4-x}$ **E** $f(x) = -x^3$

13 For a function with rule $y = \dfrac{-2}{(x+3)^3} - 5$, the maximal domain and range are

A $x \ne 3,\ y \ne -5$ **B** $x \ne -5,\ y \ne -3$ **C** $x \ne -3,\ y \ne -5$

D $x \ne -2,\ y \ne -5$ **E** $x \ne -3,\ y \ne 5$

14 A function with rule $f(x) = \frac{1}{x^4}$ can be defined on different domains. Which one of the following does not give the correct range for the given domain?

A dom $f = [-1, -0.5]$, ran $f = [1, 16]$

B dom $f = [-0.5, 0.5] \setminus \{0\}$, ran $f = [16, \infty)$

C dom $f = (-0.5, 0.5) \setminus \{0\}$, ran $f = (16, \infty)$

D dom $f = [-0.5, 1] \setminus \{0\}$, ran $f = [1, 16]$

E dom $f = [0.5, 1)$, ran $f = (1, 16]$

Extended-response questions

1 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 $64 per day + 25 cents per kilometre

Method 2 $89 per day with unlimited travel.

a Write a rule for each method if x kilometres per day are travelled and the cost in dollars is C_1 using method 1 and C_2 using method 2.

b Draw the graph of each, using the same axes.

c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, S, of a cube as a function of:

a the length x of an edge

b the volume V of the cube.

3 Express the area, A, of an equilateral triangle as a function of:

a the length s of each side

b the altitude h.

4 The base of a 3 m ladder leaning against a wall is x metres from the wall.

a Express the distance, d, from the top of the ladder to the ground as a function of x and sketch the graph of the function.

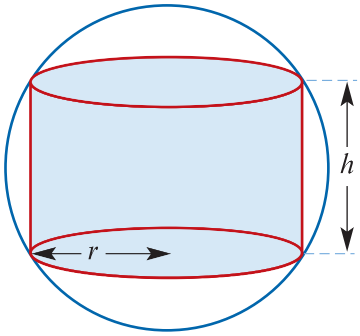
b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of x km/h. Define a function, S, which gives the average speed for the total journey as a function of x.

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.

a Define a function, V_1, which gives the volume of the cylinder as a function of its height, h. (State the rule and domain.)

b Define a function, V_2, which gives the volume of the cylinder as a function of the radius of the cylinder, r. (State the rule and domain.)



7 A function f is defined as follows:

$$f(x) = \begin{cases} x^2 - 4 & \text{for } x \in (-\infty, 2) \\ x & \text{for } x \in [2, \infty) \end{cases}$$

a Sketch the graph of f.

b Find the values of $f(-1)$ and $f(3)$.

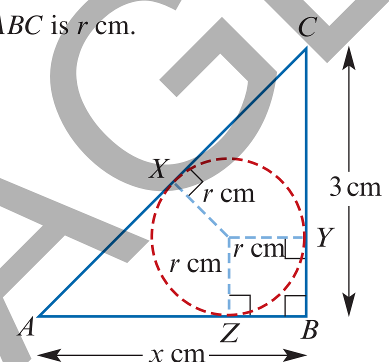
c If $h(x) = 2x$, find $f(h(x))$ and $h(f(x))$.

8 Find the rule for the area, $A(t)$, enclosed by the graph of the function

$$f(x) = \begin{cases} 3x, & 0 \le x \le 1 \\ 3, & x > 1 \end{cases}$$

the x-axis, the y-axis and the vertical line $x = t$ (for $t > 0$). State the domain and range of the function A.

9 The radius of the incircle of the right-angled triangle ABC is r cm.



a Find:

- **i** YB in terms of r
- **ii** ZB in terms of r
- **iii** AZ in terms of r and x
- **iv** CY

b Using the geometric results $CY = CX$ and $AX = AZ$, find an expression for r in terms of x.

c **i** Find r when $x = 4$.

ii Find x when $r = 0.5$.

d Use a calculator to investigate the possible values that r can take.

10 Let $f(x) = \dfrac{px + q}{x + r}$ where $x \in \mathbb{R} \setminus \{-r, r\}$.

a If $f(x) = f(-x)$ for all x, show that $f(x) = p$ for $x \in \mathbb{R} \setminus \{-r, r\}$.

b If $f(-x) = -f(x)$ for $x \neq 0$, find the rule for $f(x)$ in terms of q.

c If $p = 3$, $q = 8$ and $r = -3$, find the values of x for which $f(x) = x$.

11 **a** Let $f(x) = \dfrac{x + 1}{x - 1}$.

i Find $f(2)$, $f(f(2))$ and $f(f(f(2)))$.

ii Find $f(f(x))$.

b Let $f(x) = \dfrac{x - 3}{x + 1}$. Find $f(f(x))$ and $f(f(f(x)))$.